Mathematical Methods II
Handout 18. Power Series.

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A power series is a series of the form:

$$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + a_n(z - z_0)^n + \cdots \tag{1}$$

where z is the (complex) variable, a_i are constants called the *coefficients* of the series, and z_0 is also a constant, called the *center* of the series.

The convergence of Power Series is ruled by the following concept of *radius of convergence*:

1. Every power series converges at its center.
2. If a power series converges at the point z_1, it converges absolutely for all points in the disk of center z_0 and radius $|z_1 - z_0|$.
3. If a power series diverges at the point z_2, it diverges for all points farther from z_0 than z_2.

The radius of convergence R is that of the smallest circle that contains all the points where the series converges. No general statement can be made for points on the circle itself (cf. exercise 1).

The Cauchy–Hadamard formula allows to determine the Radius of convergence of a series from its coefficients:

$$\text{if } |a_{n+1}/a_n| \to L, \quad \text{then } R = 1/L. \tag{2}$$

For the following, with no loss of generality, we can assume the center is at $z_0 = 0$.

A power series has a unique representation, i.e., if $\sum a_i z^i = \sum b_i z^i$ for all z, then $a_i = b_i$ for all i.

* *Termwise addition or subtraction* of two power series with radius of convergence R_1 and R_2 gives a power series with radius of convergence at least $\min(R_1, R_2)$.

* *Termwise multiplication* of $f(z) = \sum a_k z^k$ and $g(z) = \sum b_m z^m$ results in the Cauchy product:

$$f(z)g(z) = \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0) z^n \tag{3}$$

that converges in the smallest radius of convergence of the two series (unproven in class).

* *Termwise Differentiation* of a power series:

$$\left(\sum_{n\geq 0} a_n z^n\right)' = \sum_{n\geq 1} n a_n z^{n-1} = \sum_{n\geq 0} (n+1) a_{n+1} z^n \,, \tag{4}$$

with the same radius of convergence.

* *Termwise Integration* of a power series:

$$\int \left(\sum_{n\geq 0} a_n z^n\right) dz = \sum_{n\geq 0} \frac{a_n}{n+1} z^{n+1} \,, \tag{5}$$

with the same radius of convergence.

We will see next lecture that Power Series are the analytic functions.

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A. Suggested readings

- "Riemann and the Cauchy–Hadamard formula for the convergence of power series", D. Laugwitz, Historia Mathematica, 21:64 (1994), at `http://goo.gl/ND2fYm`.
- "Power series", Encyclopedia of Mathematics, `http://goo.gl/1mtS7v`.

B. Exercises

1. Study the convergence of the series $\sum z^n/n$ on its radius of convergence.
2. Find the radius of convergence of the following series:

$$\sum_{n=0}^{\infty} 4^n (z+1)^n \tag{6}$$

$$\sum_{n=0}^{\infty} \frac{n^n}{n!} z^n \tag{7}$$

$$\sum_{n=0}^{\infty} \frac{n(n-1)}{2^n} (z-2i)^n \tag{8}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{z}{2\pi}\right)^{2n+1} \tag{9}$$

$$\sum_{n=0}^{\infty} \frac{n}{3^n} (z+2i)^{2n} \tag{10}$$

$$\sum_{n=1}^{\infty} \frac{5^n}{n(n+1)} z^n \tag{11}$$

$$\sum_{n=k}^{\infty} \binom{n}{k} \left(\frac{z}{2}\right)^n \tag{12}$$

C. Problems

1. Show that:

$$(1-z)^{-2} = \sum_{n\geq 0} (n+1) z^n \,,$$

   i) by using the Cauchy product, *ii*) by differentiating another series.
2. Using $(1+z)^p(1+z)^q = (1+z)^{p+q}$, show that:

$$\sum_{n=0}^{r} \binom{p}{n}\binom{q}{r-n} = \binom{p+q}{r} \tag{13}$$