Mathematical Methods II
Handout 16. Some consequences of Cauchy's theorem.

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Cauchy's inequality. On a circle of radius r circling the pole z_0, with $M = \max|f|$ on the circle: $|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}$.

Liouville's theorem. If an holomorphic function on the whole complex plane (such a function is called "entire") is bounded on $\mathbf{C}$, then $f = \text{cste}$.

Morera's theorem. If f is continuous in a simply connected domain and if $\oint_C f(z)\,dz = 0$ for every closed path in D, then f is holomorphic in D.

Fundamental theorem of algebra. A polynomial equation

$$P(z) = \sum_{i=1}^{N} \alpha_i z^i = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_N z^N \tag{1}$$

with $\alpha_i \in \mathbf{C}$ and $\alpha_N \neq 0$ has exactly n roots in $\mathbf{C}$ (possibly degenerate).

Proof: We prove first that there is at least one root: if $P(z)$ would never be zero, then $1/P(z)$ would be entire and since $|1/P(z)|$ is bounded (it tends to zero to infinity), from Liouville's theorem, it should be constant, in contradiction with the fact that $\alpha_N \neq 0$. Therefore, $P(z)$ has at least one zero.

Let's call z_0 this root. Then

$$P(z) = P(z) - P(z_0) = \sum_{i=0}^{N} \alpha_i (z^i - z_0^i) = (z - z_0)Q(z) \tag{2}$$

with Q a polynomial of degree $N-1$. Repeating the same reasoning shows that Q has at least one zero (possibly z_0 again).

We have already stated the following when studying harmonic functions. We can now prove these statements:

Gauss' mean value theorem. An analytic function is equal to the average of its values on any surrounding circle.

Proof: From Cauchy's integral formula $f(a) = \frac{1}{2\pi i}\oint_C f(z)/(z-a)\,dz$ on $C = \{z : |z-a| = r\}$, we find $f(a) = \frac{1}{2\pi i}\int_0^{2\pi} f(a + re^{i\theta}) ire^{i\theta}/(re^{i\theta})\,d\theta = \int_0^{2\pi} f(a + re^{i\theta}) rd\theta/(2\pi r)$.

Extremum modulus theorem. An analytic function in a region $\mathcal{R}$ reaches the maximum and the minimum of its modulus on the boundary of $\mathcal{R}$.

We conclude with another projection of complex results into the real space:

If f is holomorphic on a circle C of radius R and for every $z = re^{i\theta}$ inside it:

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - rRe^{i(\phi-\theta)})f(Re^{i\phi})}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\,d\phi\,. \tag{3}$$

from which we obtain, decomposing f between its real and imaginary parts $f = u + iv$, the Poisson's integral formulas on the disk:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)u(R,\phi)}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\,d\phi\,, \tag{4}$$

and also the counterpart with $u \leftrightarrow v$. This specifies completely harmonic functions inside a circle from the values on its boundary. This is used for solving the two-dimensional Laplace equation with boundary conditions given on the unit disc.

The countepart on the half plane reads, for f analytic on the upper half-plane ($y \geq 0$) containing a point $\zeta = \xi + i\eta$:

$$f(\zeta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta f(x)}{(x - \xi)^2 + \eta^2}\,dx \tag{5}$$

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whose real and imaginary parts $f(\xi+i\eta)=u(\xi,\eta)+iv(\xi,\eta)$ provide:

$$u(\xi,\eta)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\eta u(x,0)}{(x-\xi)^2+\eta^2}\,dx\,. \tag{6}$$

And also the same relation with $u\leftrightarrow v$. These are known as Poisson's integral formulas for the half-plane.

For instance, the harmonic function $u(x,y)$ defined on the half-plane (positive y) such that $u(x,0)=1$ for $-1\le x\le 1$ and zero otherwise is:

$$u(x,y)=\frac{1}{\pi}\int_{-1}^{1}\frac{y dt}{(t-x)^2+y^2}=\frac{1}{\pi}\arctan\left(\frac{y}{x-t}\right)\Bigg|_{-1}^{1}=\frac{1}{\pi}\arctan\left(\frac{y}{x-1}\right)-\frac{1}{\pi}\arctan\left(\frac{y}{x+1}\right)\,. \tag{7}$$

A counterpart, the "Schwarz integral formula" (not given) comes back to the complex plane, it allows to recover an holomorphic function, up to an imaginary constant, from the boundary values of its real part.

A. Suggested readings

- "Schwarz and Poisson formulas" on PlanetMath.org at `http://planetmath.org/encyclopedia/PoissonFormula.html` (or `http://goo.gl/aDpOL`).
- "One way of looking at Cauchy's theorem", Timothy Gowers' weblog at `http://gowers.wordpress.com/2007/09/19/one-way-of-looking-at-cauchys-theorem` (or `http://goo.gl/GXkQO`).

B. Exercises

1. Explicit the fundamental theorem of algebra on a particular example, for instance $f(z)=z^4-z^2-2z+2$.
2. Give the power expansion of $f(x)=\sin(x)$ for $x\in\mathbf{R}$. Show that sin is bounded. How does the fact that we have here an analytic function which is bounded but not constant fit with Liouville's theorem?
3. Compute $\int_0^{2\pi}\frac{(R^2-r^2)}{R^2-2Rr\cos(\theta-\phi)+r^2}\,d\phi$.
4. Compute $\int_0^{2\pi}\sin^2(\pi/6+2e^{i\theta})\,d\theta$.
5. Use Poisson's integral formula for the upper half-plane to show that:

$$u(x,y)=e^{-y}\cos(x)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\cos(t)}{(x-t)^2+y^2}\,dt\,. \tag{8}$$

6. Find the function $u(x,y)$ that is harmonic in the upper half-plane $y>0$ and which takes on the boundary values $u(x,0)=x$ for $-1<x<1$ and 0 otherwise.

C. Problems

1. Prove the extrememum modulus theorem (prove the "maximum", deduce the "minimum").
2. Prove the Poisson's integral transforms for the half-plane.

D. Correction to the Home Exam #2/7

1. *Mathematical reasoning*

Proposition: If an entire function is bounded, it is constant. In symbols: Let f be an entire function. If there exists M such that $|f(z)|\le M$ for all $z\in\mathbf{C}$, then there exists $\zeta\in\mathbf{C}$ such that $f(z)=\zeta$ for all $z\in\mathbf{C}$.

If f and g are two entire function, i.e., holomorphic everywhere, and such that $|f|<|g|$, then $h=f/g$ is also entire (indeed, it is derivable as the quotient of two functions everywhere derivable; note that g is nowhere zero since $|f|\ge 0$ and $|g|>|f|$, therefore $|g|>0$). Since $|h|$ is bounded by 1 (from $|h|=|f/g|=|f|/|g|$ and the hypothesis that $|f|<|g|$), as en entire function, it is constant according to the proposition, i.e., there exists α such that $h=\alpha$, i.e., $f=\alpha g$.

This means that entire functions are intertwined: there is no function that is always above another one, unless they are trivially related (by a constant). Entire functions get above each others and get arbitrarily high in modulus.