Mathematical Methods II
Lecture 2: Complex functions of complex numbers

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There are a lot of different and complementary ways to represent complex-valued functions of a complex variables which in full generality require a 4D space since $z \to f(z)$ with $z \in \mathbf{C}$ maps to $(x, y) \to u(x, y) + iv(x, y)$ with $(x, y) \in \mathbf{R}^2$.

The mapping of characteristic structures (such as a grid of lines) by a function, known as conformal mapping (for its property of preserving angles locally) is an insightful way to represent some (the most important) complex functions:

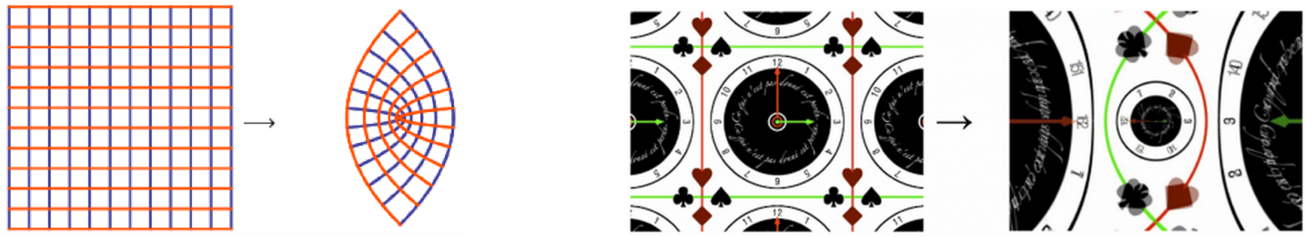

FIG. 1: Conformal mapping of gridlines by the function $z \to z^2$ mapping a simple pattern (left) and a complex one (right).

When defining complex functions, care must be taken with possible multi-valuedness, since there is no way to define consistently a function that has multiple possible values. For intance, $(z = re^{i\theta}) \to \sqrt{r}e^{i\theta/2}$, which behaves like a square root locally, is ill-defined on a loop that circle the origin.

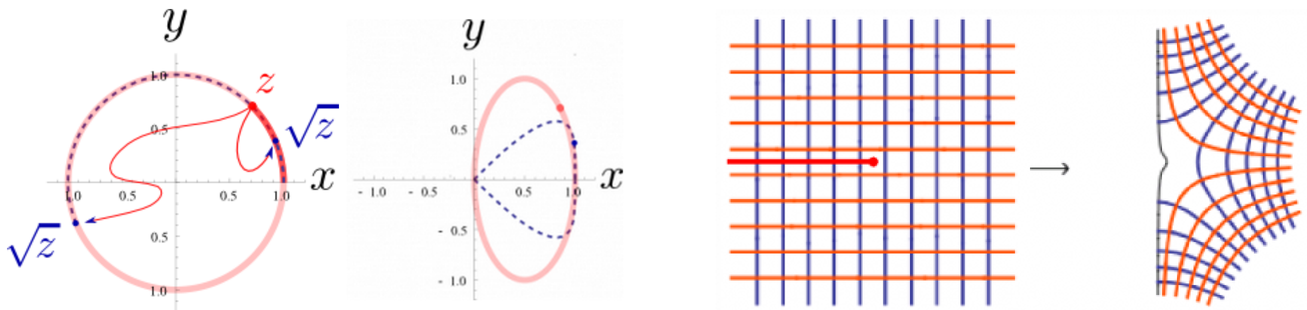

FIG. 2: By going around the origin, the function $\sqrt{r}e^{i\theta/2}$ becomes multi-valued (left). By avoiding the phase singularity, no problem is encountered (middle). Therefore, we can cut the plan with a branch cut (right).

This is due to branch points in the plane, that can be circumvented through branch cuts that forbide to circle round them and raise multi-valuedness issues (in the case of the square root, there is only one branch point, at the origin). A "principal value" can be defined on such a restricted complex plane. Full generality can be retained by defining the function on a Riemann surface that deforms the complex plane to provide a fitting source space to the function.

Such features that appear to be technical annoyances turn out to be central features of the fabric of complex numbers, which we will see in later lectures to be connected to some of their most powerful properties.

A. Suggested readings

- "*Geometry of logarithms, powers, and roots*", Chap. 5 of "The Road to Reality", Penrose (2004).

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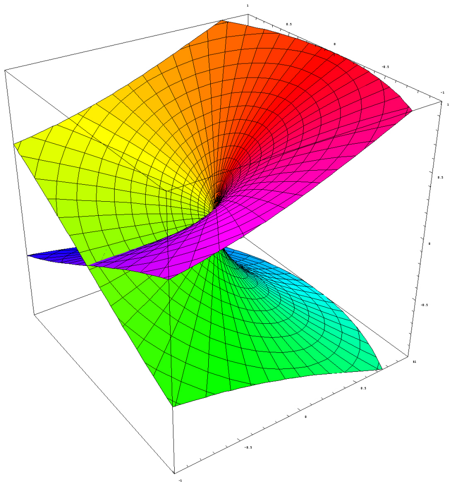

FIG. 3: Riemann surface for the square root function. This is the domain of definition of the function that allows it to be well (single-valued) defined everywhere with no restriction. The branch cut marks the region where one goes from one Riemann sheet to the other.

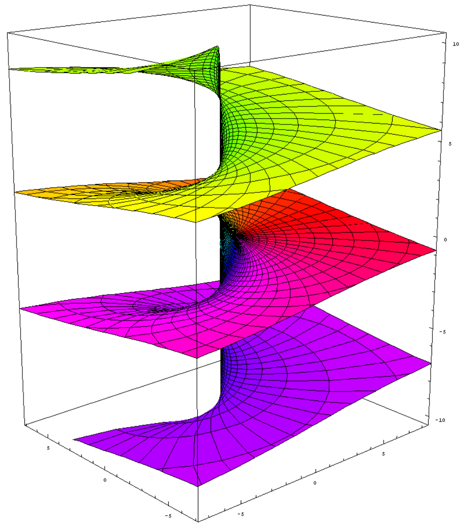

FIG. 4: Riemann surface for the logarithm; it has an infinite number of sheets.

- `http://acko.net/blog/how-to-fold-a-julia-fractal` or `http://goo.gl/sAfaI` (requires WebGL).
- "Applications conformes", Christian Mercat `http://images.math.cnrs.fr/Applications-conformes.html` or `http://goo.gl/gJIaQ` (in French).
- "Visualizing complex analytic functions using domain coloring",Hans Lundmark, `http://www.mai.liu.se/~halun/complex/domain_coloring-unicode.html` or `http://goo.gl/L27Wc`.
- "Complex Functions as Transformations", Chap. 2 of "Visual Complex Analysis", Needham, Oxford University Press (1998).
- `http://laussy.org/wiki/MMII`

B. Exercises

1. Study the conformal mapping of x^n ($n \in \mathbf{N}$) for various curves of your choice.
2. Study the conformal mapping of $1/z$, $\cos(z)$ and $\sin(z)$, $\exp(z)$ and $\log(z)$.
3. For $z = x + iy$, show that the principal value of the square root can be defined by $\sqrt{z} = \left(\sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} + i\mathrm{s}(y)\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}\right)$ with $\mathrm{s}(y) = \frac{y}{|y|}$ if $y \neq 0$ and 0 otherwise.
4. A theme of complex calculus is to derive real-calculus results from the wider and more power complex workframe. Use Euler's identity to derive expressions for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$.

C. Problems

1. Study the vizualization problem of rational functions, i.e., of the type $P(z)/Q(z)$ where P and Q are polynomials.
2. Express the complex inverse trigonometric functions arcsin, arccos and arctan as function of the complex logarithm.
3. Study the Riemann surface of $\sqrt{(z-a)(z-b)}$ for two complex values a and b.
4. For a given c, study numerically the set of $z \in \mathbf{C}$ such that the iterations of $z^2 + c$ remain bounded in the complex plane (e.g., $c = -0.8 + 0.156i$).