Mathematical Methods II
Handout 20. Taylor Series.

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(Dated: March 24, 2014)

An "*analytic* function" is one given by a convergent power series in an open disc: $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$. If $z_0 = 0$, such a Taylor series is called a Maclaurin series. The evaluation of the coefficients c_n is obtained easily by differentiating term by term. The N-th order expansion is given by *Taylor's theorem*:

$$f(z) = \sum_{k=0}^{N} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k + \frac{(z - z_0)^{N+1}}{2i\pi} \oint_{\mathcal{C}} \frac{f(w)}{(w - z_0)^{N+1}(w - z)} dw \,. \tag{1}$$

The last term $R_N(z)$ is the remainder and converges to zero as $N \to \infty$.

In the real space, Taylor Series have some disagreeable features. For instance, the function $f(x) = \exp(-1/x^2)$ is infinitely differentiable at $x = 0$ but each derivative at this point is zero (see home exam), therefore the Taylor series does not converge to its function. Another perplexing case of the real variable is given by the function $f(x) = 1/(1+x^2)$ that is everywhere defined, is bounded and is always positive. Its Maclaurin series is tedious but in principle straightforward to derive: $1/(1 + x^2) = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{k=0}^{\infty}(-1)^k x^{2k}$. It converges only for $-1 < x < 1$ since it has two accumulations points (-1 and 1) at $x = \pm 1$ and therefore has no limit. It seems strange that the series diverges at $x = \pm 1$ where the function has no problem (it is equal to $1/2$), or beyond where the function is equally well defined. In fact, one can find a Taylor series that extends beyond the interval $]-1, 1[$, for instance by expanding about $x_0 = 1/2$:

$$f\left(\frac{1}{2} + x\right) = \frac{1}{1 + (1/2)^2} + \left(-\frac{16}{25}\right) x + \left(-\frac{32}{125}\right) \frac{x}{2} + \left(\frac{2304}{625}\right) \frac{x^3}{3!} + \cdots \tag{2}$$

which provides $f\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{328}{625} \approx 0.52$ (next term is $1/2$ up to three digits after the comma). One can even compute for some points past the previous limit, e.g., for $x = \frac{1}{2} + \frac{1}{10}$, we would find, for increasing orders of the expansion:

$$f_N(1 + 1/10) \in \{0.8, 0.42, 0.37, 0.50, 0.452, 0.438, 0.459, 0.453, \dots\} \tag{3}$$

converging towards $1/(1 + [1 + 1/10]^2) \approx 0.4524$. The convergence is slow and gets slower until point $\frac{1+\sqrt{5}}{2}$ where it diverges again. All this is understood by shifting to the complex plane. The Taylor series defines a power series that one can evaluate for complex arguments. For instance $1/(1 + (i/2)^2) = \frac{4}{3}$ is indeed the limit of $1 - (i/2)^2 + (i/2)^4 - (i/2)^6 + \cdots = \sum_{k=0}^{\infty} 1/(2^{2k}) = 1/(1 - 1/4) = 4/3$.

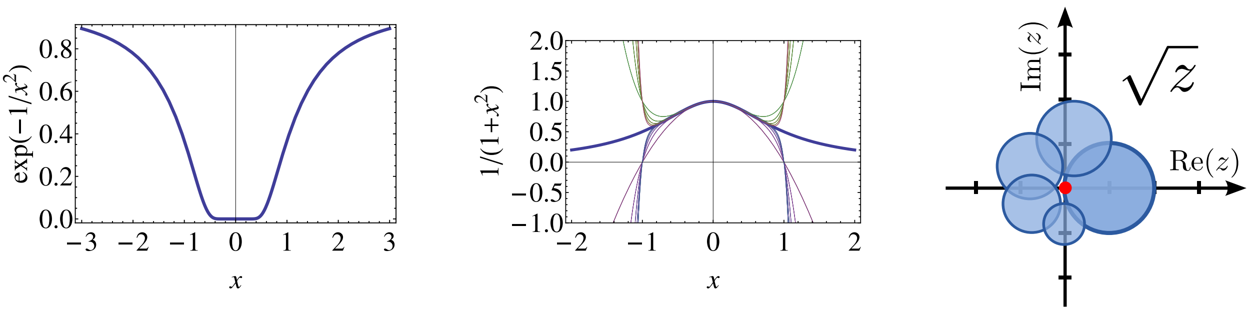

FIG. 1: (left) A smooth but non-analytic function that cannot be expanded in a Taylor series, (middle) the Taylor approximation to $1/(1 + x^2)$ on the real line and (right) the principle of analytic continuation.

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It is now apparent what is the problem with the Taylor series on the real axis. The reason lies on the imaginary axis, where the function has two poles at $\pm i$. Given the way a power series converges within its radius of convergence, the divergence beyond ± 1 is required to allow that at $\pm i$. The ability to define a Taylor series around a different centre allows a so-called process of "*analytic continuation*". The limit of the expansion about 1/2 is simply that defined by the new radius of convergence around this point, which is $|\frac{1}{2} - (\pm i)| = \sqrt{5}/2$. One can patch the complex plane with overlapping circles that go round divergences and reconstruct the entire analytic function from its definition on a neighborhood only.

Calculation of Taylor series are usually made on the basis of a handful of particular cases that should be known, through combinations, substitutions and other tricks like integration and derivation. In other cases, or if unsure, one can also use the definition Eq. (1). For instance, $f(z) = \frac{1}{1-z} \Rightarrow f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}} \Rightarrow f^{(n)}(0) = n!$ leads us to:

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k\,. \tag{4}$$

The Binomial series $f(z) = (1+z)^\nu = \sum_{k=0}^{\infty} \binom{\nu}{k} z^k$ is defined in terms of the generalized binomial coefficients $\binom{\nu}{k} = \nu(\nu-1)(\nu-2)\cdots(\nu-k+1)/k!$. From this, we can compute, for instance, $\sqrt{1+z} = 1+z/2-z^2/8+z^3/16+\cdots$ or $1/(1+z)^m = 1-mz+m(m+1)z^2/2-m(m+1)(m+2)z^3/6+\cdots$, etc. The series $1/(1+z^2)$ which we have considered at length previously arises straightforwardly from the geometric series by substitution $z \to -z^2$. A calculation by integration gives the Maclaurin series of $f(z) = \arctan(z)$ from the fact that $f'(z) = 1/(1+z^2) = \sum_{k=0}^{\infty}(-1)^k z^{2k}$. Integrating termwise: $\arctan(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} z^{2k+1} = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots$

A. Suggested readings

- `http://en.wikipedia.org/wiki/List_of_mathematical_series`.
- "Analytic Continuation", T. Needham, Visual Complex Analysis. Clarendon Press, pp. 247-257, 2000.
- "Analytic Continuation", Kevin Brown, at `http://www.mathpages.com/home/kmath649/kmath649.htm` (or `http://goo.gl/RKOie5`).

B. Exercises

1. Find the Maclaurin series and the radius of convergence of $\sin(2z^2)$, $1/(2+z^4)$ and $\cos^2(z/2)$.
2. Find the Maclaurin series and the radius of convergence of $\int_0^z \exp(-\zeta^2/2)\,d\zeta$ and $\int_0^z \sin(\zeta^2)\,d\zeta$.
3. Find the Taylor series of $1/(1-z)$ and its radius of convergence for expansion around 0, i and -1.
4. Find the Taylor series of $1/(z+i)^2$ and its radius of convergence for expansion around i.
5. Find the Maclaurin series of $1/\sqrt{4-z}$ and its radius of convergence.

C. Problem

1. Through the Taylor expansion of $1/\sqrt{1-z^2}$ and termwise integration, show that:

$$\arcsin(z) = z + \left(\frac{1}{2}\right)\frac{z^3}{3} + \left(\frac{1\times 3}{2\times 4}\right)\frac{z^5}{5} + \left(\frac{1\times 3\times 5}{2\times 4\times 6}\right)\frac{z^7}{7} + \cdots \tag{5}$$

2. Continue analytically the square root of z from the unit circle centered at $z_0 = 1$ in a path around the origin to overlap again with the initial domain (cf. Figure).

D. Home Exam (to return by 7th of April)

In the following, we will strive for rigor and clarity of the text. You can return your work collectively, if it is to be identical; however see it as a chance to write down your own solution. Each question is worth 10/3 points.

I. Show that the Maclaurin series of the function $f(x) = \exp(-1/x^2)$ does not converge to the function itself.
II. Show that integrating a power series term by term yields a power series with the same radius of convergence.
III. Compute $\sum_{k=0}^{\infty} \cos(n\theta)/2^n$.