Mathematical Methods II
Handout 14: Line and contour integrals.

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We introduce the partial derivative operators of the complex variable z and z^*:

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\quad\text{and}\quad\frac{\partial}{\partial z^*}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\tag{1}$$

Properties of the ∂_z and ∂_{z^*} operators are:

$$\partial_z f(z_0)=f'(z_0)\quad\text{and}\quad\partial_{z^*}f(z_0)=0\,.\tag{2}$$

The latter equation is the Cauchy-Riemann condition.

With these, Green's theorem reads:

$$\oint_{\mathcal{C}} f(z,z^*)\,dz=2i\iint_{\mathcal{S}}\frac{\partial f}{\partial z^*}dx\,dy\tag{3}$$

This shows in particular that if we call A the area of the domain enclosed by a contour $\mathcal{C}$:

$$A=\frac{1}{2i}\oint_{\mathcal{C}} z^* dz\,.\tag{4}$$

The complex form of Green theorem also brings us to an important theorem that we will meet over and over again:

Cauchy theorem: If f is holomorphic (or equivalently, analytic) in a region $\mathcal{R}$ and also on the contour $\mathcal{C}$ enclosing it, then:

$$\oint_{\mathcal{C}} f(z)\,dz=0\,.\tag{5}$$

As a consequence of the Cauchy theorem, if z_0 and z_1 are two complex points in a simply connected region where f is holomorphic, then $\int_{z_0}^{z_1}f(z)\,dz$ is independent of the path (that remains within the said region) of integration, which allows such a notation.

The theory of integration and differentiation are linked through the following identity:

$$\int_{z_0}^{z_1}f(z)\,dz=F(z_1)-F(z_0)\tag{6}$$

with $F'(z)=f(z)$. As an example, $\int_0^i z\,dz=(1/2)z^2\Big|_0^i=-1/2$, as we have seen yesterday on the particular case of integrating along the y axis. The path could be completely arbitrary.

As a consequence, contour integrals over paths (in the same directions) surrounding regions where the function is analytic are identical. The integral itself could be or not zero, depending on analycity outside of the bounded regions.

In fact, let us consider the case:

$$I=\oint_{\mathcal{C}}\frac{dz}{z-a}\tag{7}$$

over a contour $\mathcal{C}$ that surrounds a. We can deform the contour into a circle C centered on a, i.e., such that $|z-a|=r$, and the integral reads:

$$\oint_C\frac{dz}{z-a}=\int_0^{2\pi}\frac{ire^{i\theta}d\theta}{re^{i\theta}}=i\int_0^{2\pi}d\theta=2i\pi\tag{8}$$

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since $z = a + re^{i\theta}$ implying $dz = ire^{i\theta}d\theta$. If $\mathcal{C}$ does not surround a, the integral is zero.

In the same way, We can also compute for any positive integer $n \neq 1$:

$$\oint_{\mathcal{C}} \frac{dz}{(z-a)^n} = \int_0^{2\pi} \frac{ire^{i\theta}d\theta}{r^n e^{in\theta}} = \frac{i}{r^{n-1}} \int_0^{2\pi} e^{i(1-n)\theta}\, d\theta = \frac{i}{r^{n-1}} \frac{1}{i(1-n)} e^{i(1-n)\theta}\Big|_0^{2\pi} = 0 \tag{9}$$

so although there is a pole in a, the contour integral still gives zero. The important result to keep constantly in mind is, for $\mathcal{C}$ surrounding a (always zero otherwise):

$$\oint_{\mathcal{C}} \frac{dz}{(z-a)^n} = \begin{cases} 2i\pi & \text{if } n = 1\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{10}$$

A. Suggested readings

- "Complex-number calculus", Chap. 7 of "the Road to reality", Penrose.
- "The Gauss-Green and Cauchy Integral Theorems", W. F. Eberlein, Am. Math. Monthly, 625, 82 (1975)
- The rest of the text pasted below is in The American Mathematical Monthly, Vol. 85, No. 4 (Apr., 1978), pp. 246-256.

B. Exercises

1. Compute $\oint_{\mathcal{C}} zdz$ and $\oint_{\mathcal{C}} z^*dz$ on $\mathcal{C}$ the disk defined by $|z+1-i| = 2$ and $\oint_{\mathcal{C}} z^{*2}dz$ on the circles $|z| = 1$ and $|z-1| = 1$.
2. Compute $\int_{\mathcal{C}} \mathrm{Re}(z)\mathrm{Im}(z)dz$ on the straight line that joins $1+i$ and (a) its opposite, (b) its conjugate.
3. Compute $\int_{\mathcal{C}} \mathrm{Im}(z)\, dz$ on the shortest path between $-1+i$ and $3+2i$.
4. Verify Green's theorem for $\oint_{\mathcal{C}}(x^2-2xy)dx + (y^2 - x^3y)dy$ with $\mathcal{C}$ the square with vertices in 0, 2, $2i+2$ and $2i$.

C. Problems

1. Show that Eq. (4) can also be written as $A = \frac{1}{4i}\oint_{\mathcal{C}}(z^*\,dz - z\,dz^*)$.
2. Show that the center of gravity of the region defined by the contour $\mathcal{C}$ with area A is given by $\frac{1}{4Ai}\oint_{\mathcal{C}} z^{*2}dz$.

WHEN IS A FUNCTION THAT SATISFIES THE CAUCHY–RIEMANN EQUATIONS ANALYTIC?

J. D. GRAY AND S. A. MORRIS

1. The Looman–Menchoff theorem—An extension of Goursat's theorem. It is well known[1] that a complex-valued function $f = u + iv$, defined and analytic on a domain D in the complex plane satisfies the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

throughout D. The standard textbooks, such as those authored by Ahlfors, Cartan, Churchill, Jameson, Knopp, Sansone and Gerretson, avoid answering the question as to whether or not the converse holds. Most instead offer the following partial converse due to Goursat **[13]**.

THEOREM 1. *If $f = u + iv$, defined on a domain D, is such that*
(i) *$\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ exist everywhere in D,*
(ii) *u, v satisfy the Cauchy–Riemann equations everywhere in D, and if further*
(iii) *f is continuous in D,*
(iv) *$\partial u/\partial x$. $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ are continuous in D,*
then f is analytic in D.

The remaining standard texts offer the stronger result:

THEOREM 2. *If $f = u + iv$, defined on a domain D, is such that*
(i) *$\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ exist everywhere in D,*
(ii) *u, v satisfy the Cauchy–Riemann equations everywhere in D, and if further*
(iii) *u, v, as functions of two real variables, are differentiable everywhere in D,*
then f is analytic in D.

Recently the authors began a search to discover precisely what is known regarding the converse.