Mathematical Methods II
Handout 22. Singularities.

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A singularity is rigorously defined in complex calculus: A point z_0 is a singularity of a function f if there is a point where f is analytic in every neighborhood of z_0 and f is not analytic at z_0. The singularity is "*isolated*" if there exist a neighborhood where the only singularity is z_0 itself.

The embodiment of a singularity is certainly $f(z) = 1/z$ at $z_0 = 0$. The function is holomorphic everywhere else (with derivative $-1/z^2$) but is not defined at the origin, where the function goes to infinity: $\lim_{z\to 0} f(z) = \infty$. Proof: for all $C \in \mathbf{R}$, there exists z_C such that $(|z| < |z_C|) \Rightarrow |f(z)| > C$. Indeed, $z_C = e^{i\theta}/C$ for any $0 \le \theta < 2\pi$ does the job. Not all singularities bring the function to infinity, however. In fact, such a singularity can be "canceled" by a numerator going to zero at least as fast as the denominator. For instance $\sin z/z$ is not defined per se at $z = 0$. Defining it to be 0 there, one get a function that is in fact everywhere analytic (it is known as the cardinal sine $\mathrm{sinc}(z)$). This is clear from its Laurent expansion $\mathrm{sinc}(z) = \frac{z-z^3/6+z^5/120+\cdots}{z} = 1 - z^2/6 + z^4/120 + \cdots$ from which we see that the cardinal sine is a cosine lookalike $1 - z^2/2 + z^4/23 + \cdots$. Such a singularity, that is apparent only as analycity can be restored or enforced by a proper choice of the value of the function at the point that seems to be the problem, is called a "*removable singularity*". That is the first (and simplest) possible type of singularity in our list of examples:

$$f_0(z) = \frac{\sin(z)}{z}\,, \quad f_1(z) = \sin\frac{1}{z}\,, \quad f_2(z) = \frac{1}{\sin z} \quad \text{and} \quad f_3(z) = \frac{1}{\sin\frac{1}{z}}\,, \tag{1}$$

For the other cases as well, the key is to use Laurent series. This gives:

$$f_1(z) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots\,, \qquad f_2(z) = \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \cdots\,, \tag{2}$$

while f_3 has no Laurent expansion. The isolated singularities are classified by the number of terms in the principal part (remember that this is the series of negative powers from the Laurent expansion). If the principal part is finite, we speak of "*poles*". The order of the singularity is the largest exponent in this finite sum. If the principal part is infinite, the singularity is "*essential*".

Thus, $\sin(1/z)$ has a simple pole (i.e., pole of order 1), $1/\sin(z)$ has an essential singularity while the singularity of $1/\sin(1/z)$ is not isolated. We will not deal with the latter type, but will show that it is indeed not isolated. This means that in any neighborhood of the origin, the function has a singularity, i.e., $\sin(1/z)$ has a zero. Phrased

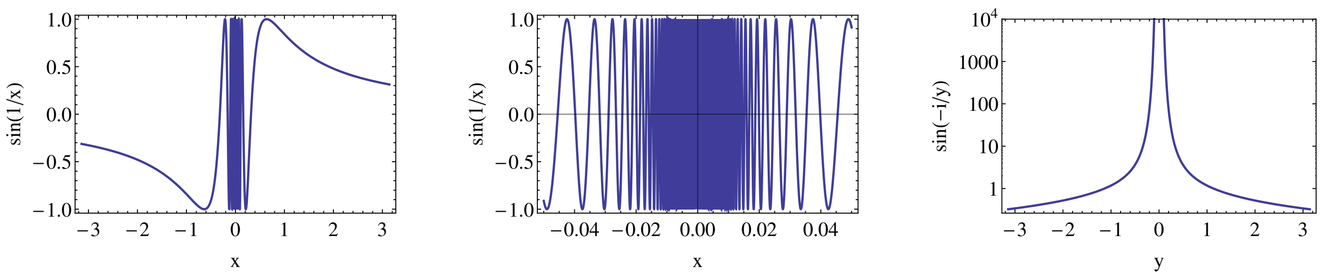

FIG. 1: The behaviour of $f_1(z) = \sin(1/z)$ on the real axis (left and middle figures) and the imaginary axis (right figure). It takes all values between -1 and 1 in the former case in any neighborhood of zero and diverges in the latter case. Picard's theorem states that all values (except possibly one) are taken when approaching the essential singularity

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equivalently, for any $\epsilon > 0$, there is at least one zero of $\sin(1/z)$ for $|z| < \epsilon$, i.e., there is at least one zero of $\sin(w)$ for $|w| > 1/\epsilon$, which is clearly true since the sine has zeros at all multiples of π.

There is an important difference between poles and essential singularities. The pole behaves as expected from a divergence: If $f(z)$ has a pole at z_0, then $|f(z)| \to \infty$ as $z \to z_0$. The proof is left as an (easy) problem. On the other hand, an essential singularity has a more complicated behaviour. It does not need diverge in the first place. For instance, $\sin(1/z)$ on the real axis is bounded (it lies in the interval $\mathcal{I} = [-1, 1]$). Its limit, if it would exist, would therefore be a finite number. There is no limit on the real axis as the function has the entire interval $\mathcal{I}$ as accumulation points. On the imaginay axis, on the other hand, $\sin(1/z) = \sin(-i/y) = -i\sinh(1/y)$, and its modulus diverges like $e^x/2$ with $x = 1/y$ as $y \to 0$. This is shown in the Figure.

The absence of a limit for an essential singularity is almost maximally violated by *Picard's theorem*, which states that: If f is analytic with an isolated essential singularity at z_0, it takes all, except possibly one, complex values in any neighborhood of z_0. The proof is a bit involved and is not given in class, but it can easily be illustrated with a particular case.

Clearly, singularities often are due to a denominator going to zero. This makes the study of zeros of interest in connection with singularities. It is then useful to know that "*the zeros of an analytic function are isolated*". The terminology for zeros is similar: a n-th order zero is such that the function is zero as well as all the $n-1$ successive derivatives. In the Taylor expansion $\sum_{k=0}^{\infty} c_k(z-z_0)^k$, the coefficient $c_k = 0$ for $k < n$ and $c_n \neq 0$. We now prove that zeros of analytic functions are isolated. Assume that the zero is of order n, then:

$$f(z) = (z - z_0) \sum_{k=n}^{\infty} a_k (z - z_0)^{n-k} \,. \tag{3}$$

Let us call $g(z)$ the function defined by the Series. It is, by definition, analytic and satisfies $g(z_0) = a_n$ (otherwise the zero would not be of order n). Now given that it is continuous (since it is derivable, being analytic), for any $\epsilon > 0$, there exists a neighborhood of z_0 such that $|g(z) - g(z_0)| < \epsilon$. By using the reverse triangle inequality:

$$\big||g(z)| - |g(z_0)|\big| \leq |g(z) - g(z_0)| \tag{4}$$

we can then conclude that $|g(z)| \geq |g(z_0)| - \epsilon$, i.e., for ϵ small enough, $|g(z)| > 0$ in a neighborhood of z_0. Since $(z - z_0)$ is zero only at z_0, this proves the assertion.

If f is analytic at z_0 and has a zero of nth order there, then for any g also analytic and such that $g(z_0) \neq 0$, we then know that $h(z) = g(z)/f(z)$ is has a pole of order nth order at z_0. Such functions that are holomorphic except for isolated points where they have poles of finite orders are called "*meromorphic*". They provide useful links to Riemann surface and in particular with the Riemann sphere.

A. Suggested readings

- `http://en.wikipedia.org/wiki/Singularity_(mathematics)` and its "See Also" section.
- "Removable Singularities, Poles, and Essential Singularities." S.G. Krantz, Handbook of Complex Variables. Birkhäuser (1999).

B. Exercises

1. Study the zeros of $(z-i)^3$, e^z, $\cosh(z)$, $\sin^4 z/2$, $\sin^2 \pi z/z^2$ and $\sin(2z)\cos(2z)$.
2. Characterize the singularities of $1/(z+2i)^2$, $1/(z+2i)^2 + 8/(z-i)^3$, $1/(e^z - e^{2z})$, $e^{1/(z-1)}/(e^z - 1)$ and $\cot^2(z)$.
3. If f is analytic and has a zero of order n at z_0, what is the order of the zero of f^2 at z_0?

C. Problems

1. Study the relationship between the cardinal sine and the cosine.
2. Prove that if f has a pole of order n at z_0, $\lim_{z \to z_0} f(z) = \infty$.
3. Study the essential singularity of $\sin(1/z)$ and $\exp(-1/z^2)$. In particular show that they take all values (except possibly one) in any neighborhood of the singularity.