Mathematical Methods II
Handout 5: Continuity (for the physicist).

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f has the limit w_0 as $z \to z_0$ provided that the value $f(z)$ can be made as close as required to the value w_0 by taking z to be sufficiently close to z_0. We write:

$$\lim_{z\to z_0} f(z) = w_0\,. \tag{1}$$

Formally:

$$(\forall\epsilon > 0)(\exists\delta > 0)(|z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon)\,. \tag{2}$$

This is the Weierstrass definition of continuity.

In the U-V representation of $f(x + iy) = u(x, y) + iv(x, y)$, for the real function u (resp. v), for which continuity reads:

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0$$

iff

$$(\forall\epsilon > 0)(\exists\delta > 0)(\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \Rightarrow |u(x, y) - u_0| < \epsilon)\,. \tag{3}$$

Theorem:

$$\lim_{z\to z_0} f(z) = w_0 \qquad \Leftrightarrow \qquad \lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0\,, \tag{4}$$

for $w_0 = u_0 + iv_0$ and $z_0 = x_0 + iy_0$.

The limit must not depend on the trajectory by which it is approached. For instance, if $U(x, y) = 2x^3/(x^2 + y^2)$, $\lim_{(x,y)\to(0,0)} U(x, y) = 0$, while $U(x, y) = (xy)/(x^2 + y^2)$ has no limit at the origin.

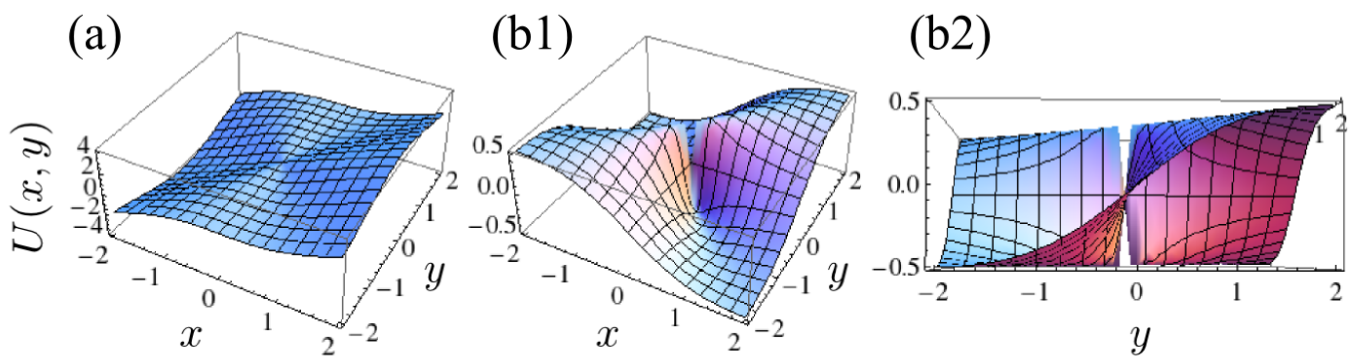

FIG. 1: 3D plots of $U(x, y)$ for (a) $U(x, y) = 2x^3/(x^2 + y^2)$ and (b) $U(x, y) = (xy)/(x^2 + y^2)$. The former has a limit at zero, the latter not. As a consequence, it is not continuous there.

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Definition: The complex function f is continuous at z_0 iff:

1. $f(z_0)$ exists.
2. $\lim_{z\to z_0} f(z)$ exists.
3. $\lim_{z\to z_0} f(z) = f(z_0)$.

A function is "*continuous*" if it is continuous at all points.

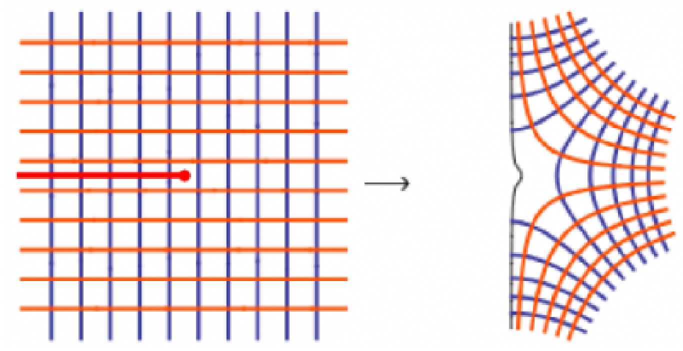

FIG. 2: Conformal mapping of gridlines by the function $z \to \sqrt{z}$. A discontinuity appears on the principal value as a result of the branch cut.

Theorem: If $f(z)$ and $g(z)$ are continuous at z_0, then the following functions are also continuous at z_0:

1. $-f(z)$,
2. $f(z) + g(z)$,
3. $f(z)g(z)$,
4. $f(z)/g(z)$, provided that $g(z_0) \neq 0$.
5. $f(g(z))$, provided that $f(z)$ is continuous in a neighborhood of $g(z_0)$.

Theorem: A complex polynomial is continuous.

A. Exercises

1. Show that the limit of a function must be unique.
2. Show that $\lim_{z\to 0} z^*/z$ does not exist.
3. Show that $f(z) = z^2$ is continuous.
4. Show that $f(z) = e^z$ is continuous.
5. Prove the theorems on continuity of the sum, product and compositions of functions.

B. Problem

Show that $f(z) = \sqrt{z}$ is continuous for all z and reconcile that with the statement of Fig. 2.