Mathematical Methods II
Handout 8: Differentiability and Cauchy-Riemann

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We study differentiability of w seen as a function of u and v:

$$w(z) = u(x, y) + iv(x, y)\,.$$

with u and v functions with a real and imaginary part of two variables x and y making up $z = x + iy$.

The Cauchy-Riemann conditions state that if w is differentiable at $z_0 = x_0 + iy_0$, then:

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)\,, \tag{1}$$

$$\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)\,. \tag{2}$$

There is an equally important counterpart for the reverse relationship, namely, if the partial derivatives of u and v exist at (x_0, y_0) and are differentiable in the sense of a function of multiple variables, then $w = u + iv$ is differentiable at $x_0 + iy_0$.

Specifically:

$$\lim_{z\to z_0} \frac{w(z) - w(z_0)}{z - z_0} = \frac{\partial w}{\partial x}(z_0) = -i\frac{\partial w}{\partial y}(z_0)$$

which proves that w is differentiable as a complex function (independently of the path of approach). We also find the value of the derivative. The last equality can also be seen as the Cauchy-Riemann equations, written more economically.

[You may find in some textbooks that continuity of the partial derivatives is required. This is in fact too strong a condition. While it is true that the sole existence of partial derivatives satisfying the Cauchy-Riemann equations is not enough to ensure complex differentiability, it is enough that u and v be real differentiable, which is a stronger condition than the existence of the partial derivatives, but is a weaker one than these partial derivatives themselves also be continuous.]

If the Cauchy-Riemann conditions holds on an entire open, the function is, by definition, holomorphic there (and therefore, also analytical).

We conclude with an insight into a more involved notion of complex derivation. Introducing $\partial w/\partial z^*$, with z^* the conjugate of z. Using the chain rule:

$$\frac{\partial w}{\partial z^*} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z^*} \tag{3}$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\frac{1}{2} + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\frac{i}{2} \tag{4}$$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{5}$$

where we have used $x = (z + z^*)/2$ and $y = (z - z^*)/2$ to compute (formally) the partial derivative of z^*. If the Cauchy-Riemann conditions hold, $\partial w/\partial z^* = 0$. This means that complex-differentiability means the function depends on z only, not z^*.

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A. Suggested readings

- "Complex number calculus", Chap. 7 of "The Road to Reality", Penrose (2004).
- "Riemann surfaces and complex mappings", Chap. 8 of "The Road to Reality", Penrose (2004).
- "When is a function that satisfies the Cauchy-Riemann equations analytic?", J. D. Gray and S. A. Morris. Amer. Math. Monthly **85** 246 (1978).

B. Exercises

1. Study the differentiability of $f(x,y)=2x+3iy^2$, $f(x,y)=\exp(-i(x^2+y^2))$, $f(x,y)=3x^2y+2^2-y^3-2y^2$, z^n.

C. Problems

1. Obtain the Cauchy-Riemann conditions in the polar representation.
2. Find a function $f(x,y)$ which has partial derivatives at a given point (x_0,y_0) that satisfy the Cauchy-Riemann conditions but that is not differentiable for the complex variable $x+iy$ at this point.

D. (Easy) Problems

1. *Calculation of a Derivative*

Show that the derivative of the logarithm is the inverse (you can use a change of variable, $w=\ln z$).

2. *Damped driven oscillator*

The dynamics of a damped harmonic oscillator reads:

$$m\ddot{x}+\gamma\dot{x}+kx+F=0$$

with m the mass, γ the decay and k the spring constant some constants $\in \mathbf{R}$, F a driving force and x the oscillator's displacement. This equation can be solved exactly for any driving force using the complex solution to the unforced equation $z(t)$ that is a combination of complex exponentials. Show this and study the solutions.

3. *Mathematical Reasoning*

Definition: We will call a function "Entire" if it is holomorphic everywhere.

Theorem: If an entire function is bounded, then it is constant.

(we will prove this theorem in class.)

1. Express the theorem logically (with symbols).
2. Let us now assume f and g two entire functions. Prove that if f and g are such that $|f|<|g|$, then there exists α such that $f=\alpha g$.
3. Discuss what this means.

(we will see in class that this remains true for the case $|f|\leq|g|$ as well; the particular case above is easy to prove.)

E. CONTINUOUS EVALUATION

1. Arrange all the (Easy) Problems of this and the previous handout in order of complexity, from what seems to you more simple to more complex.
2. Solve two of these (of your choice [you can do as many as you want]).