Mathematical Methods II
Handout 27. The Riemann and the Bloch Spheres.

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We have studied complex numbers as numbers on the plane. "Numbers" in the sense of Weil, of being objects with a structure that befits this attribute (associativity, additivity, multiplications and, importantly, commutation, which is why quaternions aren't numbers). Riemann showed how complex numbers actually live in different "superficies", e.g., the square root lives in a two-times folded manifold.

Here we will study complex numbers on a sphere, rather than on a plane. The problem of mapping a plane to a sphere (or vice-versa) is an old one, already tackled by Ptolemy, possibly also before him by the Egyptians. It is also a preoccupation for the public at large, when it comes to cartography. It is now known overall as the problem of "stereographic projection".

For complex calculus, it is customary to adopt the convention of projecting between a sphere and a plane with the later intersecting the former in its equator (see Figure).

The Riemann sphere is then defined as the set of points $S = \{Z \in \mathbf{R}^3 \, : \, |Z| = 1\}$.

The stereographic projection is the one that maps each point $z \in \mathbf{C}$ to $Z \in S$:

$$z = x + iy \quad \rightarrow \quad Z = (\xi, \eta, \zeta) \tag{1a}$$

$$\xi = \frac{2x}{x^2+y^2+1}\,, \quad \eta = \frac{2y}{x^2+y^2+1}\,, \quad \zeta = \frac{x^2+y^2-1}{x^2+y^2+1}\,, \tag{1b}$$

$$z = \frac{\xi + i\eta}{1-\zeta}\,. \tag{1c}$$

The proof is standard geometric algebra: the segment (N, z) is parametrized by:

$$(N, z) = \{\big(tx, ty, (1-t)\big) \, : \, 0 \le t \le 1\}\,, \tag{2}$$

Now a point $p(t) \in (N, z)$ intersects S iff $|p(t)|^2 = 1$, i.e., iff

$$t^2x^2 + t^2y^2 + (1-t)^2 = 1 \tag{3}$$

Gathering variables, the equation is $t^2(|z|^2+1) - 2t = 0$, with solutions $t = 0$ and $t = 2/(|z|^2+1)$.

These formulas can be inverted, yielding Eq. (1c), which can be readily found for the real and imaginary parts through similarity of the triangles NAZ and NOz: $x/1 = \xi/(1-\zeta)$ (for the real part, id. for the imaginary part).

The straight distance between points of the Riemann sphere seen as points in $\mathbf{R}^3$ provides a metric for the corresponding complex points (the so-called "chordal metric").

We remind that a *metric* is a function χ of two points such that:

1. $\chi(z_1, z_2) = \chi(z_2, z_1)\,,$

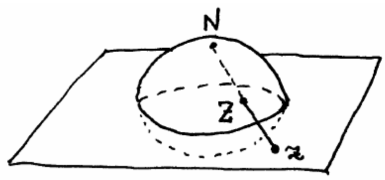

Fig. 1

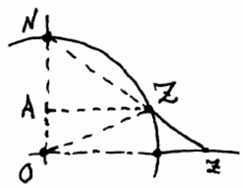

Fig. 2

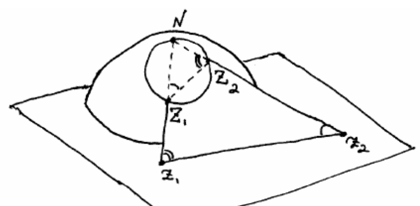

Fig. 3

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2. $\chi(z_1, z_2) \geq 0$,
3. $\chi(z_1, z_2) = 0 \Rightarrow z_1 = z_2$,
4. $\chi(z_1, z_3) \leq \chi(z_1, z_2) + \chi(z_2, z_3)$.

One can check these properties for χ as a function of z_1 and z_2, that we now express in this form. Similarity of NOz and NAZ implies:

$$|N - z| = \frac{|N - Z|}{1 - \zeta} \tag{4}$$

On the other hand, $|N - z| = \sqrt{1 + |z|^2}$ (by Pythagoras) and $1 - \zeta = 2/(1 + |z|^2)$ (by Eq. (1b)). Therefore, $|N - Z| = 2/\sqrt{1 + |z|^2}$, from which we establish:

$$|N - Z||N - z| = 2\,. \tag{5}$$

Applying this equation to two sets of points z_1, z_2 and their image Z_1, Z_2, we get: $|N-Z_1||N-z_1| = |N-Z_2||N-z_2|$, i.e.,

$$\frac{|N - Z_1|}{|N - Z_2|} = \frac{|N - z_2|}{|N - z_1|}\,. \tag{6}$$

i.e., the triangles NZ_1Z_2 and Nz_2z_1 are similar (note the order of the indices), therefore:

$$\chi(z_1, z_2) = |Z_1 - Z_2| = \frac{2|z_1 - z_2|}{\sqrt{1 + |z_1|^2}\sqrt{1 + |z_2|^2}}\,. \tag{7}$$

With this representation, we can now measure lenghts between points at infinity. This is straightforward on the sphere since this merely computes the distance to the north pole:

$$\chi(z, \infty) = \sqrt{|(\xi, \eta, \zeta) - (0, 0, 1)|^2} = \sqrt{\frac{4x^2 + 4y^2 + 4}{(x^2 + y^2 + 1)^2}} = \frac{2}{\sqrt{1 + |z|^2}}\,. \tag{8}$$

At such the Riemann sphere is an extended complex plane: the plane **C** plus one point: infinity. This simple provision allows rational functions on the complex plane to become continuous functions on the Riemann sphere, with the poles of the rational function now sent to infinity. More generally, any meromorphic function is in fact a continuous function defined on the Riemann sphere.

In Quantum Mechanics, the Riemann sphere allows to describe quantum states of the simplest quantum mechanical object: the qubit. The sphere is then known as the Bloch sphere, and represents a pure state $\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$ as the corresponding point with polar coordinates.

A. Suggested readings

- "Ptolemy's inequality, chordal metric, multiplicative metric.", M. S. Klamkin and A. Meir, Pacific J. Math., 101, 255 (1982), `http://goo.gl/JQLZr0`.
- "Quantum computation and quantum information" Cambridge University Press, Nielsen, Michael A; Chuang, Isaac L., (2010).

B. Exercises

1. Show that χ is a metric.
2. Show that $\chi(z_1, z_2) = \chi(1/z_1, 1/z_2)$.
3. Compute $\chi(z, 1/z)$.

C. Problems

1. Show that the stereographic projection is conformal.
2. Show that the stereographic projection is circle preserving.