Mathematical Methods II
Handout 9: Harmonic Functions

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An harmonic function is a solution to Laplace's equation $\nabla^2 f = 0$, where $\nabla^2 = \nabla \cdot \nabla$ (div of grad) is the Laplacian. In the complex calculus course, we will consider 2D functions, so:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\,. \tag{1}$$

From the Cauchy-Riemann equations, it follows that the real and imaginary parts of an holomorphic function are harmonic. The opposite is true if the functions are defined on a connected space.

Harmonic functions are equal to their local averages: if u is harmonic in a connected region G that contains an open ball $\mathcal{B}(z_0, r)$ for some r then:

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta\,.$$

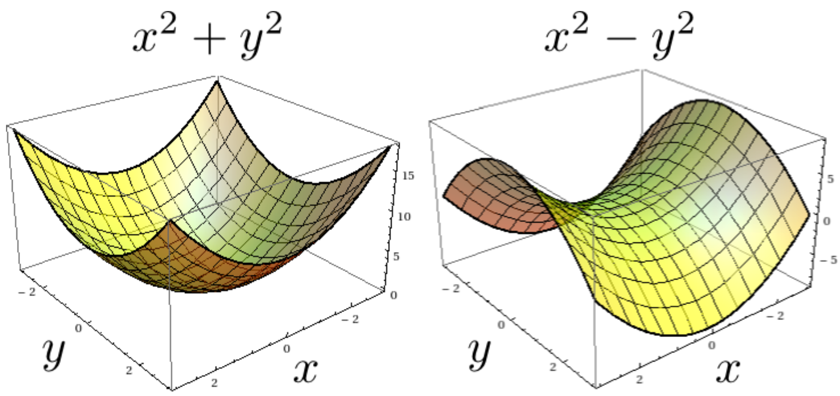

FIG. 1: The function $x^2 + y^2$ is not harmonic, having a local minimum. Its counterpart $x^2 - y^2$, having a saddle point, lacks such a local minimum (or maximum) and, being the real part of z^2, is in fact harmonic.

As a corollary, an harmonic function has no local maxima or minima (or, more precisely, if it has, it is constant).

As another corollary, if f is harmonic in a bounded connected region, then its absolute maximum and minimum occur on the boundary.

Finally, if f and g are harmonic functions such that $f = g$ on the boundary of the domain where they are defined, then $f = g$ also inside the entire region as well.

Therefore, a solution to Laplace's equation is uniquely determined if:

1. the value of the function is specified on all boundaries (Dirichlet boundary conditions), or
2. the normal derivative of the function is specified on all boundaries (Neumann boundary conditions).

Because Laplace's equation is linear, the superposition of any two solutions is also a solution.

In the Cartesian geometry on a simple support, Laplace equation can be solved by the method of separation of variable: the "assumption" $u(x, y) = X(x)Y(y)$ turns $\nabla^2 u$ into:

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2}\,, \tag{2}$$

which can be solved by conventional methods (try!)

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A. Suggested readings

- "On the Relation of an Analytic Function of z to Its Real and Imaginary Parts" L. M. Milne-Thomson, The Mathematical Gazette, **21** 228 (1937)
- "Recovering Holomorphic Functions from Their Real or Imaginary Parts without the Cauchy-Riemann Equations", W. T. Shaw, SIAM Review, **46** 717 (2004)
- "Short Proofs of Three Theorems on Harmonic Functions" H. P. Boas and R. P. Boas, Proceedings of the American Mathematical Society, **102** 906 (1988)

B. Exercises

1. Study harmonic functions of your choice (in particular play with their 3D representation as a function of $x, y \in \mathbf{R}$, cf. Fig. 1).
2. Show that $u(x, y) = e^x \sin(y)$ is harmonic. Find the holomorphic function of which u is the real part.
3. Show that $\log(\sqrt{x^2 + y^2})$ and $\arctan(y/x)$ are harmonic.
4. Suppose u and v are harmonic, and $c \in \mathbf{R}$. Prove that $u + cv$ is also harmonic.

C. Problem

1. *The Milne-Thomson method of finding the harmonic conjugate of a function:* Given the harmonic function $u(x, y)$, construct

$$v(x, y) = \mathrm{Im}\left[2u\left(\frac{x + iy}{2}, \frac{x + iy}{2i}\right)\right] . \tag{3}$$

   Baring exceptions where the method fails, v is the harmonic conjugage of u and $u + iv$ is holomorphic. The method works when the analytic function is a power series centered about the origin. Check an study this method.

D. Off the beaten track

Points & questions raised after the class:

- On the Stereographic function transforming circles into circles: `http://ams.org/samplings/feature-column/fc-current.cgi` (or `http://goo.gl/w2exm3`).
- *Complex Numbers as Residue Classes of Polynomials mod* $(x^2 + 1)$, Rosemary Schmalz, The Two-Year College Mathematics Journal, Vol. **3**, No. 2 (Autumn, 1972), pp. 78-80, `http://www.jstor.org/stable/3026924`.