Mathematical Methods II
Lecture 1: Introducing Complex Numbers

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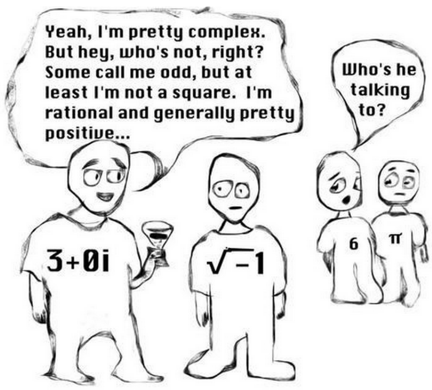
Historically, complex numbers ("numeros complejos") are a trick to extend the domain of application of algebraic formulas.



By defining a number i such that:

$$i^2 = -1\,, \tag{1}$$

we open a new algebra (addition, multiplication, etc. . .) of the numbers of the type $x + iy$ with x and y "normal" (real) numbers, keeping the usual rules. Therefore, for $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)\,, \tag{2a}$$
$$z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)\,, \tag{2b}$$
$$z_1/z_2 = z_1z_2/|z_2|^2\,, \tag{2c}$$

where we have used the important operation of complex conjugation $z^* = x - iy$ that brings back complex numbers to the real space: $|z|^2 = zz^* = x^2 + y^2$.

With such rules, we can derive the all-important Euler formula:

$$\exp(ix) = \cos(x) + i\sin(x)\,, \tag{3}$$

which also links with the polar representation of complex numbers, and allows us to compute all "complex expressions" by extending the familiar elementary algebra to complex numbers, such as:

$$2^i\,,\quad \ln i\,,\quad i^i\,,\quad \sqrt{i}\,,\quad \cos(i)\,,\quad \text{Etc.} \tag{4}$$

SUGGESTED READINGS

- "Algebra", Chap. 22 of "The Feynman Lectures on Physics", Vol. 1, Feynman *et al.*, Addison Weisley (1970).
- "Magical complex numbers", Chap. 4 of "The Road to Reality", Penrose (2004).

PROFESSORS

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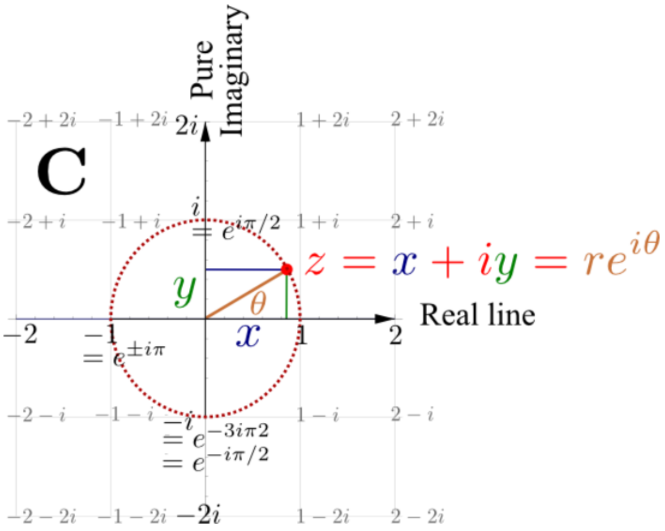

FIG. 1: The Complex plane (Argand space).

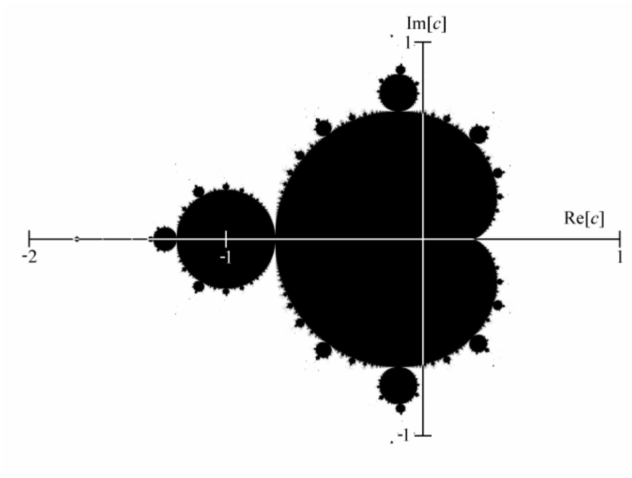

FIG. 2: The Mandelbrot set.

EXERCISES

1. Compute $1 + i + i^2$, $(3 + i)^2$, $(2 + i)^3$.
2. Play with Eq. (3) using $\exp(z) = \sum_{n=0}^{\infty} z^n/n!$ and studying its real and imaginary part.
3. Calculate all expressions of Eq. 4 and also i^{12345}, π^i, $(i^i)^i$ and $i^{(i^i)}$.
4. Calculate $(x + iy)^n$ for $n \in \mathbf{N}$. Explore De Moivres formula: $e^{in\theta} = (\cos\theta + i\sin\theta)^n$.

PROBLEMS

1. Calculate the area of the Mandelbrot set.
2. Calculate $i!$.

SCHEDULE FOR THE NEXT FEW SESSIONS

21.01: Complex functions of complex numbers.
22.01: Exponentials, trigonometric functions, hyperbolics and their inverses.
23.01: Representations: Stereographic projection, Riemann sphere, Bloch sphere.
27.01: Limits and continuity (for the Physicist).
3.02: Limits and continuity (for the Mathematician).
4.02: Derivatives and analyticity.
10.02: Differentiability and Cauchy-Riemann.
11.02: Harmonic functions and Laplace equation.
17.02: Complex Potentials.
18.02: More on conformal mapping.
24.02: Integrals in the complex plane.
25.02: Line and contour integrals.
3.03: The Cauchy-Goursat theorem and its integral forms.
4.03: Consequences of the Cauchy theorem.