MATHEMATICAL METHODS II
Lecture 7: Derivative and analyticity

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(Dated: February 4, 2014)

Much like for a real function, the "derivative" at $z_0 \in \mathbf{C}$ of a complex function f is the limit, if it exists:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \,.$$

The fact that the derivative must be the same along any direction through which the limit is approached result in much stronger properties than for the real variable. For this reason, we will qualify of *holomorphic* at z_0 any function which has a derivative for all points in a neighborhood of z_0.

More generally, we will otherwise say that a function is differentiable when it merely has a derivative (say, on a single point but not compulsorily on all those in an open containing this point).

With the above definition, one can compute the complex derivatives of "traditional" functions, e.g., for $n \in \mathbf{N}$:

$$(z^n)' = nz^{n-1} \,,$$

or $(e^z)' = e^z$, $(\cos(z))' = -\sin(z)$, $(\ln z)' = 1/z$, etc. All these results are the same for the case of the real variable. This is because these function are *analytic*, i.e., they can be expressed as an infinite series of the type:

$$f(z) = \sum_{n=0}^{\infty} a_n \left(z - z_0\right)^n \tag{1}$$

where $a_n = f^{(n)}(z_0)/n!$.

In the complex variable, the link between analyticity and differentiability is very strong: *analytic functions are holomorphic, and vice-versa.* In particular, if the first derivative exists in a neighborhood of a point, so do all higher order derivatives. We cannot yet prove nor even quite appreciate the depth of this equivalence between the two concepts; this will come in the following of the course.

A. Suggested readings

- "Infinitely-differentiable function that is not analytic", Ariel Scolnicov, PlanetMath.org at `http://planetmath.org/InfinitelyDifferentiableFunctionThatIsNotAnalytic.html` (or `http://goo.gl/lqpIU`).

B. Exercises

1. Compute the complex derivatives of $\sqrt{z}$, $\exp(-z^2)$, a^z (for $a \in \mathbf{C}$) and $(1+z)/(1-z)$.
2. Show that if f and g are differentiable, then $(f+g)' = f' + g'$, $(f/g)' = (f'g - fg')/g^2$, $(fg)' = f'g + fg'$ and $(f \circ g)' = (f' \circ g)g'$. You don't have to do all, try the first one and carry on until you get bored.
3. Study the differentiability of $f(z) = \mathrm{Re}(z)$, $\mathrm{Im}(z)$, $|z|^2$, $|z|^4$, $f(x+iy) = x^2 + iy^3$ and $f(x,y) = x^4 - 6x^2y^2 + y^4 + i(4x^3y - 4xy^3)$.

C. Optional Exercises

1. Consider the function $f(x) = \exp(-1/x)$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$. Study the analyticity of this function (of a real variable x).
2. Study the analyticity of the function $1/(z^2+1)$ both on the real line ($z = x$) and for the complex variable z.

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D. (Easy) Problems

1. Group theory

An nth root of unity, where $n \in \mathbf{N}$ is a positive integer, is a complex number z satisfying the equation:

$$z^n = 1\,. \tag{2}$$

Show that the nth roots of unity form under multiplication a cyclic group of order n. Represent the solutions graphically.

Reminder: A group is a set G together with an operation $\cdot$ (called the "group law" of G) that combines any two elements a and b of G to form another element, denoted $a \cdot b$ or simply ab. To qualify as a group, the set and operation $(G, \cdot)$ must satisfy four requirements known as the "group axioms":

1. *Closure*: For all a, $b \in G$, the result of the operation, $a \cdot b$ is also in G.
2. *Associativity*: For all a, b and $c \in G$:
$$(a \cdot b) \cdot c = a \cdot (b \cdot c)\,.$$
3. *Identity*: There exists an element $e \in G$ such that for every element $a \in G$, the equation $e \cdot a = a \cdot e$. Such an element is unique (and is called the "identity").
4. *Inverse*: For each $a \in G$, there exists an element $b \in G$ such that $a \cdot b = b \cdot a = e$.

The group is cyclic when all elements are generated by a single one (the generator) through repeated applications of the operation.

2. Coupled oscillators

Two harmonic oscillators a and b with free energies ω_a and ω_b and linearly coupled with coupling strength g have the Hamiltonian:

$$H = \begin{pmatrix} \omega_a & g \\ g & \omega_b \end{pmatrix}\,. \tag{3}$$

Find the energies of the normal modes as a function of the detuning $\omega_a - \omega_b$. When the oscillators suffer dissipation, at the rates γ_a and γ_b, their energies become complex: $\omega_a + i\gamma_a$ and $\omega_b + i\gamma_b$. Study the coupling in this condition (energies and dissipation of the normal modes), again as function of detuning, and also as function of the dissipation.

3. Mathematical reasoning

A Cauchy sequence is a sequence of numbers (z_n) such that for every $\epsilon > 0$, there exists an integer $N \in \mathbf{N}$ such that for all integers $m, n > N$:

$$|z_m - z_n| < \epsilon\,.$$

Show that a convergent sequence is a Cauchy sequence. Conversely, not all Cauchy sequences converge (when this is the case, the space is called "complete"). Show that the open ball $\mathcal{B}(0,1)$ in the complex plane is not complete. Show that the set of rational numbers is not complete either. Can you find more examples?

I. PROJECT (OPTIONAL)

Study with a computer the iteration of:

$$z_{n+1} = (|\mathrm{Re}(z_n)| + i|\mathrm{Im}(z_n)|)^2 + c \tag{4}$$

with, say, $z_0 = 0$ and $c \in \mathbf{C}$.