Mathematical Methods II
Handout 15. The Cauchy-Goursat theorem and its integral forms.

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A more general proof of Cauchy's theorem than that based on Green's theorem (that assumes continuity of the derivatives) is due to Goursat. On a triangle ABC cut through its mid-points (E between AB, F between BC and D between AC): $\oint_{ABCA} = \oint_{DAED} + \oint_{EBFE} + \oint_{FCDF} + \oint_{DEFD}$ scaling down the problem from one triangle to four sub-triangles. Through iterations $\left|\oint_\Delta f(z)\,dz\right| \leq 4^n \left|\oint_{\Delta_n} f(z)\,dz\right|$ with Δ the initial trajectory $ABCD$ and Δ_n that of the nth iterated triangle obtained by taking successive mid-points of Δ_{n-1} ($\Delta_0 = \Delta$). With P the perimeter of Δ, assuming differentiability of f, we arrive at $|\oint_\Delta f(z)ds| \leq \epsilon P^2$ for any $\epsilon > 0$, i.e., the contour integral is zero.

The main consequences of Cauchy's theorem is Cauchy's integral theorem: an holomorphic function f in a simply connected domain D satisfy, for all $z_0 \in D$ and any simple closed path $\mathcal{C}$ that encloses z_0:

$$f(z_0) = \frac{1}{2i\pi}\oint_\mathcal{C} \frac{f(z)}{z - z_0}\,dz\,. \tag{1}$$

The Cauchy's integral formula also proves that holomorphic are infinitely differentiable:

$$f'(z_0) = \frac{1}{2i\pi}\oint_\mathcal{C} \frac{f(z)}{(z-z_0)^2}\,dz\,, \quad f''(z_0) = \frac{2!}{2i\pi}\oint_\mathcal{C} \frac{f(z)}{(z-z_0)^3}\,dz\,, \quad \cdots \quad f^{(n)}(z_0) = \frac{n!}{2i\pi}\oint_\mathcal{C} \frac{f(z)}{(z-z_0)^{n+1}}\,dz\,. \tag{2}$$

A. Suggested readings

- `http://en.wikipedia.org/wiki/Cauchy's_integral_formula`.
- "A Simple Proof of the Fundamental Cauchy-Goursat Theorem", E. H. Moore, Trans. Amer. Math. Soc. **1**, 499 (1900) doi:10.2307/1986368 or `http://goo.gl/ZJXoa`.

B. Exercises

1. Show that f being analytic is a sufficient but not necessary condition for $\oint f(z)dz = 0$.
2. Compute the following integrals on a path that encloses the singularity in each case:

$$\oint_\mathcal{C} \frac{\sin z}{z}\,dz\,, \quad \oint_\mathcal{C} \frac{z^3 - 6}{2z - i}\,dz\,, \quad \oint_\mathcal{C} \frac{e^{2z}}{\pi z - i}\,dz\,, \quad \oint_\mathcal{C} \frac{z^2 \sin z}{4z - 1}\,dz \quad \text{and} \quad \oint_\mathcal{C} \frac{e^z}{ze^z - 2iz}\,dz\,.$$

3. Compute $\oint_\mathcal{C} \frac{z^2+1}{z^2-1}\,dz$ on $\mathcal{C} = \{z : |z-1| = 0\}$, $\mathcal{C} = \{z : |z+1| = 0\}$, $\mathcal{C} = \{z : |z| = 1/2\}$ and $\mathcal{C} = \{z : |z| = 2\}$.
4. Compute for $\mathcal{C} = \{z : |z| = 3/2\}$:

$$\oint_\mathcal{C} \frac{z^4 - 3z^2 + 6}{(z+i)^3}\,dz \quad \text{and} \quad \oint_\mathcal{C} \frac{\exp(z)}{(z-1)^2(z^2+4)}\,dz\,.$$

C. Problems

- Check that the Cauchy-Goursat demonstration fails for z^*.
- Prove Eq. (1) by deforming $\mathcal{C}$ to a circle centered on z_0 and the change of variable $z - z_0 = \epsilon e^{i\theta}$ for a circle of vanishing radius.

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D. Correction to the Home Exam #2/7

1. *Group theory*

The numbers z such that $z^n = 1$, are those, in polar form, $(re^{i\theta})^n = r^n e^{in\theta}$ such that $|r|^n = 1$ (that is, $r = 1$ since $r \in \mathbf{R}$) and $n\theta = 2k\pi$ with $k \in \mathbf{N}$, since we are looking for *all* the solutions, not only one to define a principal value (so we allow ourselves to wind around the phase as many times as possible). The solutions are $\theta = 2k\pi/n$ with $k = 0, 1, \ldots n-1$, since $k = n$ yields $\theta = 2\pi$ which is the same solution as $\theta = 0$. The nth roots of unity are therefore:

$$\mathcal{S} = \{1, e^{i\frac{2\pi}{n}}, e^{i\frac{4\pi}{n}}, \ldots, e^{i\frac{2k\pi}{n}}, \ldots, e^{i\frac{2(n-1)\pi}{n}}\} \tag{3}$$

and are n elements in total, distributed regularly on the unit circle, starting at $z = 1$ on the real axis. We will now show that $(\mathcal{S}, \times)$ is a cyclic group (this being done, we have already shown it is of order n).

A generic element of $\mathcal{S}$ is of the form $e^{i\frac{2k\pi}{n}}$ for an integer k such that $0 \leq k \leq n-1$.

1. *Closure:* assume $a, b \in \mathcal{S}$; there is therefore κ, λ integers in $[0, n-1]$ such that $a = e^{i\frac{2\kappa\pi}{n}}$ and $b = e^{i\frac{2\lambda\pi}{n}}$. Their product $ab = e^{i\frac{2(\kappa+\lambda)\pi}{n}}$ is not trivially in $\mathcal{S}$ according to the definiton since $0 \leq \kappa+\lambda \leq 2(n-1)$ (that is, it is in $[0, 2(n-1)]$ instead of $[0, n-1]$). If $\kappa + \lambda \leq n-1$, $ab \in \mathcal{S}$. If on the other hand, $n-1 < \kappa+\lambda \leq 2(n-1)$, we then define $\eta = (\kappa+\lambda) - n$, that leads to $ab = e^{i\frac{2(\kappa+\lambda)\pi}{n}} = e^{i\frac{2\eta\pi}{n}} e^{i\frac{2n\pi}{n}} = e^{i\frac{2\eta\pi}{n}}$, that is in $\mathcal{G}$ since $-1 < \eta \leq n-1$ and η being the sum of two integers is also an integer.
2. *Associativity:* this property in $\mathcal{S}$ is trivially inherited from the associativity of complex numbers.
3. *Identity:* The identity in $\mathcal{S}$ is the same as $\mathbf{C}$, namely, $1 = \exp^{0\times 2i\pi/n}$ which is trivially such that $1 \times a = a \times 1 = a$ for all $a \in \mathcal{S}$.
4. *Inverse:* For all $a \in \mathcal{S}$, that is, for $e^{i2\pi k/n}$ for $0 \leq k \leq n-1$, the inverse in $\mathbf{C}$ of a is $e^{-i2\pi k/n}$, which again is not trivially in $\mathcal{S}$, since $-k$ is not in $[0, n-1]$. We can either prove that this number is in fact in $\mathcal{S}$ or, more rapidly, it is enough to explicit the inverse of a, namely, $b = e^{i2\pi(n-k)/n}$ for $k \neq 0$, that satisfies $ab = ba = 1$ and is in $\mathcal{S}$ since $1 \leq k \leq n-1$ implies $-1 \geq -k \geq 1-n$ and $n-1 \geq n-k \geq 1$. If $k = 0$, the element is 1 which is its own inverse and therefore also in $\mathcal{S}$.

The generator of this cyclic group is $e^{i2\pi/n}$, whose kth power is $e^{i2k\pi/n}$ and thus indeed the kth element of the group.

2. *Coupled Oscillators*

We will call $\Delta = \omega_a - \omega_b$ the detuning. The normal modes being the eigenvalues of H, we find:

$$\omega_\pm = \frac{\omega_a + \omega_b}{2} \pm \sqrt{g^2 + \Delta^2/4}\,. \tag{4}$$

This shows that the normal modes have the average energy plus the square root correction that comes from the coupling. We write the average energy as $\omega_a - \Delta/2$ and since energy is defined up to a constant, we set $\omega_a = 0$ to find:

$$\omega_\pm = \delta \pm \sqrt{g^2 + \delta^2}\,, \tag{5}$$

where we wrote $\delta = \Delta/2$. Plotting these solutions show the characteristic anticrossing of energy levels due to coupling.

This was the pure Hamiltonian result. To model dissipation, we consider the Hamiltonian:

$$H = \begin{pmatrix} \omega_a + i\gamma_a & g \\ g & \omega_b + i\gamma_b \end{pmatrix} \tag{6}$$

whose eigenvalues read:

$$\tilde{\omega}_\pm = \frac{\omega_a + \omega_b}{2} + i\frac{\gamma_a + \gamma_b}{2} \pm \sqrt{g^2 - (\gamma_b - \gamma_a + i\Delta)^2/4}\,. \tag{7}$$

This shows that not only the energy but also the dissipation (imaginary part of the energy) gets averaged. The coupling term is interesting. When $\gamma_a = \gamma_b$, the coupling recovers the previous form (of no dissipation). Therefore not absence of dissipation but its imbalance, is detrimental to the coupling. This is more clear at resonance, where the coupling term reads $\sqrt{g^2 - (\gamma_a - \gamma_b)^2/4}$. When:

$$g > |\gamma_a - \gamma_b|/2 \tag{8}$$

the square root becomes complex and the anticrossing vanishes: the system is in "weak-coupling". This simple result shows that when coupling dominates, the oscillators have same dissipation but different energies, while when dissipation dominates, they have same energy but different "broadening" (term for width of their energy spread as a result of dissipation). This is a central theme of quantum light-matter interactions.

3. *Mathematical reasoning*

Proposition: A convergent sequence is a Cauchy sequence.

Proof: Let (z_n) be a convergent sequence, i.e., $(\exists\zeta \in \mathbf{C})(\forall\epsilon > 0)(\exists N \in \mathbf{N})(n > N) \Rightarrow (|z_n - \zeta| < \epsilon/2)$. Since we can bound $|z_m - z_n| = |(z_m - \zeta) - (z_n - \zeta)| \leq |z_m - \zeta| + |z_n - \zeta| \leq \epsilon$ for all $m, n \geq N$, it is a Cauchy sequence.

The open ball is not complete since a sequence such as $z_n = 1 - 1/n$ is a Cauchy sequence (for all $\epsilon > 0$, there exists $N = \lceil 2/\epsilon \rceil$ such that $|z_n - z_m| = |\frac{1}{m} - \frac{1}{n}| \leq \frac{1}{m} + \frac{1}{n} \leq \frac{2}{N} \leq \epsilon$). However it does not converge in the open ball since the limit, $\lim_{n\to\infty} z_n = 1$, is not in the set (there is no $n \in \mathbf{N}$ such that $1 - 1/n = 1$).

The set $\mathbf{Q}$ is also incomplete: any irrational number can be approached by a series of rationals (say the decimal expansion) which is Cauchy but that, by definition, does not converge in $\mathbf{Q}$. In fact the real numbers are the "completion" of $\mathbf{Q}$.

4. *Calculation of a Derivative*

To compute $(\ln(z))'$, we introduce $w = \ln z$ with the idea that:

$$(\ln(z_0))' = \lim_{z\to z_0} \frac{\ln(z) - \ln(z_0)}{z - z_0} \tag{9a}$$

$$= \lim_{z\to z_0} \frac{w - w_0}{e^w - e^{w_0}} \tag{9b}$$

$$= \lim_{w\to w_0} \frac{w - w_0}{e^w - e^{w_0}} \tag{9c}$$

$$= \lim_{w\to w_0} \frac{1}{\frac{e^w - e^{w_0}}{w - w_0}} \tag{9d}$$

$$= \frac{1}{\lim\limits_{w\to w_0} \frac{e^w - e^{w_0}}{w - w_0}} \tag{9e}$$

$$= \frac{1}{(e^w)'|_{w_0}} = \frac{1}{e^{w_0}} = \frac{1}{z_0}\,. \tag{9f}$$

There are two steps that would need to be justified, namely, Eq. (9c) where the limit is changed from z to w and Eq. (9e) where the order of the limit and the inverse is interchanged. Formally, this means demonstrating:

Proposition 1: If $\lim_{z\to z_0} g(z) = g(z_0)$ then $\lim_{z\to z_0} f(g(z)) = \lim_{g(z)\to g(z_0)} f(g(z))$.

Proposition 2: If $\lim_{z\to z_0} f(z) = f(z_0)$ then $\lim_{z\to z_0}(1/f(z)) = 1/f(z_0)$, i.e., $1/\big(\lim_{z\to z_0} f(z)\big)$.

This is here left as an exercise.

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Other solutions will be provided in tomorrow's handout.