MÉTODOS MATEMÁTICOS II
Handout 10: Complex Potentials.

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Potential theory studies functions Φ that satisfy Laplace's equation $\nabla^2\Phi = 0$. In two dimensions, there is a tight link with complex analysis: for such a Φ, there exists F holomorphic such that $F(z) = \Phi(x, y) + i\Psi(x, y)$ with Ψ also harmonic. The function F is called the "*complex potential*" of Φ. Ψ has the physical meaning of "lines of forces", since the curves $\Psi(x, y) = \text{cst}$ are orthogonal to the equi-potential lines $\Phi(x, y) = \text{cst}$.

For example, the potential between two parallel plates situated on the x axis at α and β and with an applied potential Φ_α and Φ_β gives rise to the real potential:

$$F(z) = \left(\frac{\Phi_\alpha - \Phi_\beta}{\alpha - \beta}\right) z - \frac{\Phi_\alpha \beta - \Phi_\beta \alpha}{\alpha - \beta}\,, \tag{1}$$

The equipotentials are lines horizontal to the plates, $x = \text{cst}$, while lines of forces are those perpendicular, joining them, $y = \text{cst}$.

Another example in polar representation. The Laplacian in such coordinates gives rise to the complex potential:

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} \Rightarrow F(z) = \frac{\Phi_\alpha - \Phi_\beta}{\ln(r_\alpha - r_\beta)}\ln(z) + \frac{1}{2}\frac{\Phi_\alpha + \Phi_\beta}{\Phi_\alpha - \Phi_\beta}\frac{\ln(r_\alpha r_\beta)}{\ln(r_\alpha/r_\beta)} \tag{2}$$

with equipotentials $r = \text{cst}$ circles between the cylinders and lines of forces $\theta = \text{cst}$ the perpendiculars joining them.

A strength of potential theory is that any linear superposition of solutions is also a solution. The complex potential of one line charge at point z_0 being $F_1(z) = \alpha\ln(z - z_0)$, that for two such lines, say of opposite charges α at z_0 and $-z_0$ reads:

$$F(z) = \alpha[\ln(z - z_0) - \ln(z + z_0)] \tag{3}$$

The equipotentials are then $\text{Re}(F(z)) = \text{cst}$, i.e.,

$$\left|\frac{z - z_0}{z + z_0}\right| = \text{cst} \tag{4}$$

These are circles. The lines of force are given by $\text{Im}(F) = \arg\frac{z-z_0}{z+z_0} = \text{cst}$, which reduces to $\theta_1 - \theta_2 = \text{cst}$ which are also circles (exercise).

The notion of a complex potential applies to any 2D potential, such as electric field, temperature for the heat equation, etc. We consider further applications in fluid mechanics, namely, for a 2D steady, non-viscuous and incompressible fluid, which allows the fluid velocity to derive from a potential:

$$V_x = \frac{\partial\Phi}{\partial x}\,, \quad V_y = \frac{\partial\Phi}{\partial y}\,. \tag{5}$$

The holomorphic $F(z) = \Phi + i\Psi$ provides (from $F'(z) = \partial_x\Phi - i\partial_y\Psi$) the complex velocity as:

$$V_x + iV_y = F'(z)^*\,. \tag{6}$$

The speed at a given point is $|F'(z)|$. Points where $F'(z) = 0$ are "stationary points", where the fluid doesn't move.

The complex potential z^2 describes flow past a corner. The potential

$$F(z) = z + \frac{1}{z}\,. \tag{7}$$

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yields, passing by the polar form, $F = (r + \frac{1}{r})\cos\theta + i(r - \frac{1}{r})\sin\theta$, streamlines as:

$$\left(r - \frac{1}{r}\right)\sin\theta = \text{cst} \tag{8}$$

which are shown in Fig. 1, right panel. This describes the important problem of a flow past a cylinder.

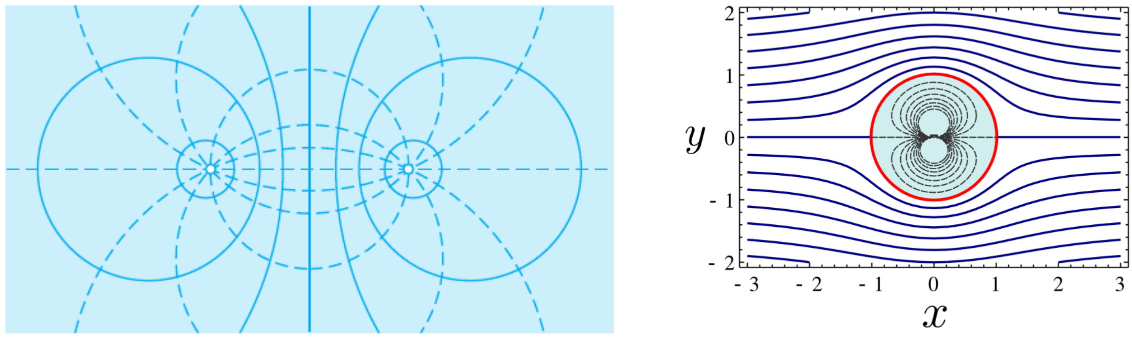

FIG. 1: Left: Complex potential from two lines perpendicular to the plane, giving rise to equipotentials (solid) and lines of force (dashed); Right: Potential flow from the complex potential $z + 1/z$. The solution $r^2 = 1$ is used as the obstacle, the outer lines represent the flow of an ideal fluid past it, the inner (dashed) lines are solutions not used here.

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A. Suggested readings

- "2D potential flow" Chap. 6 of "Fluid Mechanics", R. Fitzpatrick, p. 101 at `http://goo.gl/zy8aj`.
- "The Laplacian in polar coordinates", Zhi Lin, at `http://goo.gl/TvNto`.

B. Exercises

1. Check that the potential of Eq. (3) gives rise to that show in Fig. 1.
2. Study the complex potential $F(z) = iz^3$.

C. Problems

1. Find the temperature field around a long thin wire of radius $r_1 = 1$mm that is electrically heated to $T_1 = 100°$ and is surrounded by a circular cylinder of radius $r_2 = 100$mm, kept at temperature $T_2 = 20°$.
2. Show that $F(z) = \arccos z$ defines the potential of a slit. How about arccosh?

D. Evaluation

The final note will be composed of 30% continuous evaluation, 35% partial exam and 35% the final exam.

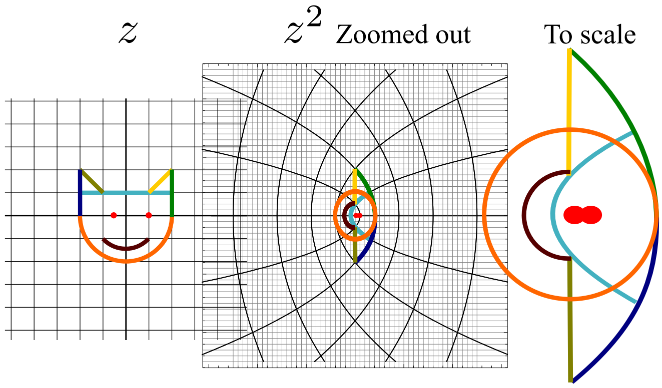
E. Solution of Continuous Evaluation (#1/7)

- $i^{12345} = i^{4\times3086+1} = (i^4)^{3086} i = i$ since $i^4 = 1$.
- $\ln(i) = \ln(e^{i\pi/2}) = i\pi/2$ so $\ln(\ln(i)) = i\pi/2 + \ln(\pi/2)$.
- The stages cycle between $i+1$, $(i+1)/2$, 1 and repeat. As there are six stages, the result is 1.

This Arnold cat can be decomposed into horizontal and vertical segments as well as rays and arc of circles.

Lines $x = c$ transform into $u = c^2 - v^2/(4c^2)$ parabolas and lines $y = c$ lines into $u = v^2/(4c^2) - c^2$ parabolas.

Rays with angle θ transform into rays with angle 2θ and arc circles $\theta_1 \leq \theta \leq \theta_2$ into arc circles $2\theta_1 \leq \theta \leq 2\theta_2$.



Complex numbers z such that their additive inverse $(-z)$ and multiplicative inverse $(1/z)$ are the same are those such that $-z = 1/z$, i.e., $z^2 = -1$. The solution of this equation is $z = \pm i$. Indeed, $-(\pm i) = \mp i = 1/(\pm i)$.