MÉTODOS MATEMÁTICOS IV
Lecture 4: Limit and Continuity

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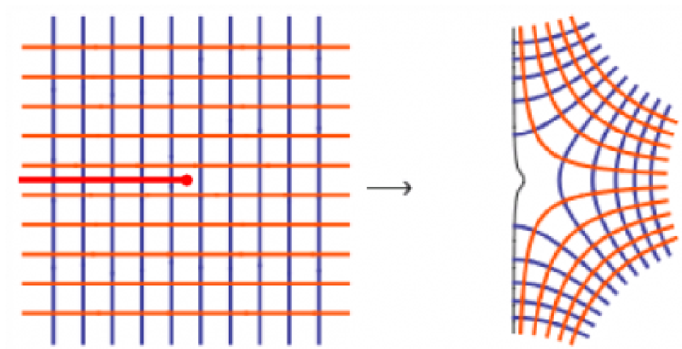

FIG. 1: Conformal mapping of gridlines by the function $z \to \sqrt{z}$. A discontinuity appears on the principal value as a result of the branch cut.

We will take the advantage of the question of continuity of complex functions to approach the subject from a more Mathematical viewpoint than is usual for a physicist, with an inclination to rigorous proofs such as those that a Mathematician would desire.

Definition: A topology on a set X is a pair $(X, \mathcal{J})$ with $\mathcal{J}$ a subset of $\mathcal{P}(X)$ containing at least $\emptyset$ and X and which is closed under the formation of arbitrary unions and finite intersections.

The members of $\mathcal{J}$ are called open sets.

Definition: A function f is continuous if the inverse image of every open set is open.

This is the exact Mathematical definition. We will focus on the particular case of metric spaces. Here it is enough to deal with "open balls":

Definition: An "open ball" $B(a, r)$ of radius r centered on a is the set of points $B(a, r) = \{z \in \mathbf{C} \mid |z, a| < r\}$.

An open set is any union of open balls.

Definition: A neighbourhood $\mathcal{V}$ of a point z is a set such that there exists an open set O that contains z and is included in $\mathcal{V}$.

Proposition: A function f is continuous at the point z iff for any neighborhood $\mathcal{V}$ of $f(x)$, there is a neighborhood $\mathcal{U}$ of x such that $f(\mathcal{U}) \subset \mathcal{V}$.

Note that "iff" stands of "if an only if", or, in symbols, $\Leftrightarrow$. Applied to open balls:

$$(\forall \epsilon > 0)(\exists \delta > 0)(|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon)\,.$$

A funtion f is continuous if it is continuous everywhere. In logical symbols:

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$$(\forall z \in \mathbf{C})(\forall \varepsilon > 0)(\exists \eta > 0)(\forall w \in \mathbf{C})(|z - w| < \eta \;\Rightarrow\; |f(z) - f(w)| < \varepsilon)\,.$$

This is the Weierstrass definition of continuity. Note that the exact positioning of the closes (parentheses) and their individual statements are important. In fact, the variation:

$$(\forall \varepsilon > 0)(\exists \eta > 0)(\forall z \in \mathbf{C})(\forall w \in \mathbf{C})(|z - w| < \eta \;\Rightarrow\; |f(z) - f(w)| < \varepsilon)$$

describes a stronger version of continuity, known as "uniform continuity". These are the version we will be working with. They rely on limiting procedures, so we remind these as well in the concept of complex variables:

Definition: The sequence of points z_n converge to z_0, and we write $\lim\limits_{n\to\infty} z_n = z_0$ iff:

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N})(\forall n \in \mathbf{N})(n > N \Rightarrow |z_n - z_0| < \epsilon)$$

In this context, f is continuous at z_0 iff $\lim_{z\to z_0} f(z) = f(z_0)$.

A. Suggested readings

- Browse `http://en.wikipedia.org/wiki/List_of_general_topology_topics` (http://goo.gl/BHv3n) and see how many Mathematical concepts of topology you are now familiar with; explore the others.
- "Principles of Mathematical Analysis", 3rd Edition, Rudin, McGraw-Hill (1976).

B. Exercises

1. Show that the limit of a function must be unique.
2. Show that $\lim_{z\to 0} z^*/z$ does not exist.
3. Show that $f(z) = z^2$ is continuous.
4. Show that $f(z) = e^z$ is continuous.
5. Show that $f(z) = \sqrt{z}$ is continuous.

C. Problems

1. Prove that the topological and Weierstrass definition of continuity are equivalent.
2. Prove the theorems on continuity of the sum, product and compositions of functions.