Mathematical Methods II
Handout 26. Summation of Series.

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To remind us some of the key results of the last lectures, after the Easter break, we will study an application of Residue theory into the problem of computing series of numbers, such as

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots=\frac{\pi^2}{6}\,. \tag{1}$$

The generic formulas to do so are:

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_{z_0\in Z(f)} \mathrm{Res}_{z=z_0}\pi\cot(\pi z)f(z)\,, \tag{2a}$$

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\sum_{z_0\in Z(f)} \mathrm{Res}_{z=z_0}\pi\csc(\pi z)f(z)\,, \tag{2b}$$

$$\sum_{n=-\infty}^{\infty} f\left(\frac{2n+1}{2}\right) = -\sum_{z_0\in Z(f)} \mathrm{Res}_{z=z_0}\pi\tan(\pi z)f(z)\,, \tag{2c}$$

$$\sum_{n=-\infty}^{\infty} (-1)^n f\left(\frac{2n+1}{2}\right) = -\sum_{z_0\in Z(f)} \mathrm{Res}_{z=z_0}\pi\sec(\pi z)f(z)\,, \tag{2d}$$

where $Z(f)$ is the set of poles of f, assumed to have simple poles only, that do not make the series diverge, and which is such that:

$$|f(z)|\le\frac{M}{|z|^k}\,. \tag{3}$$

We provide an example of this method:

$$\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}=\frac{\pi}{2a}\coth(\pi a-\frac{1}{2a^2})\,. \tag{4}$$

Indeed, with $f(z)=1/(z^2+a^2)=1/[(z-ia)(z+ia)]$, the related series $\sum_{-\infty}^{\infty}$ is minus the sum of the residues of $\pi\cot(\pi z)/(z^2+a^2)$ at the poles $\pm ia$. We find:

$$\mathrm{Res}_{z=ia}\frac{\pi\cot(\pi z)}{z^2+a^2}=\lim_{z\to ia}\frac{\pi\cot(\pi z)}{z+ia}=\frac{\pi\cot(i\pi a)}{2ia}=-\frac{\pi\coth(\pi a)}{2a} \tag{5}$$

which is the same for the other pole. From this follows straightforwardly Eq. (4).

Mittag-Leffler theorem: The Mittag-Leffler theorem provides a meromorphic function with prescribed poles (it is the counterpart of Weierstrass factorization theorem that does the same for holomorphic functions with prescribed zeros). Given the poles a_i, $1\le i\le N$, with residues b_i, the Mittag-Leffler theorem provides the Series expansion:

$$f(z)=f(0)+\sum_{n=1}^{\infty}b_n\left[\frac{1}{z-a_n}+\frac{1}{a_n}\right]\,. \tag{6}$$

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A. Suggested readings

- "The Mittag–Leffler Theorem: The Origin, Evolution, and Reception of a Mathematical Result, 1876-1884", L. E. Turner, Historia Mathematica, **40**, 36, 2013 `http://goo.gl/5LsLpj`.

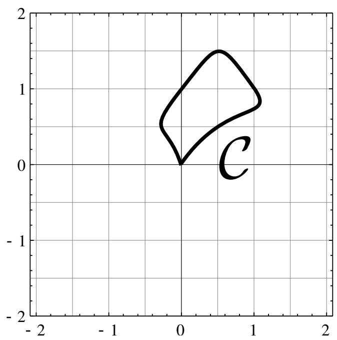
B. Exercises

1. Check Eq. (1).
2. Compute $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$
3. Compute $\sum_{n=-\infty}^{\infty}(-1)^n/(n+a)^2$.
4. Show that:

$$\cot z = \frac{1}{z} + 2z\sum_n\left(\frac{1}{z^2-\pi^2} + \frac{1}{z^2-4\pi^2} + \cdots\right) . \tag{7}$$

C. Problems

1. Prove the other equalities in Eq. (2a); pay attention to the case of infinite poles for f.
2. Prove the Mittag-Leffler theorem.



Partial Exam (recovery)

For $\mathcal{C}$ the closed trajectory in the complex plane shown in the figure:

1. What is the image $\mathcal{C}'$ of $\mathcal{C}$ by the transform $z \to iz$? (1pt)
2. What is the image $\mathcal{C}''$ of $\mathcal{C}$ by the transform $z \to \frac{1}{z}$? (1pt)
3. What is $\oint \frac{\sin(z)}{4z^2 + 2z(1-i) - i}dz$ on $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{C}''$? (3pts)
4. Show that the limit of a sequence is a point of accumulation for this sequence. (2pts)
5. Show that if z_1 and z_2 are two points of accumulation of a sequence that are also a limit for this sequence, then $z_1 = z_2$. (3pts)

Solution

For questions 1 and 2, one can take a list of particular points, such as $\{0, (1+i)/2, 1+3i/4, 1+i, (1+3i)/2, i, -(1/4)+i/2\}$, and compute their images. The image of $\mathcal{C}$ can be obtained by interpolation.

1.The multiplication by i adds a phase $\pi/2$ so that $\mathcal{C}'$ is the rotating of $\mathcal{C}$ by $90°$.

2. The inverse of a point $1/(x+iy) = (x-iy)/(x^2+y^2)$ is the conjugate (symmetric image for the x axis) with inverse module. The image (see figure) is therefore fully in the lower quadrant, with a gap from the origin and extending towards infinity.

3. We can use Cauchy's integral formula (or the residues). First is to locate the poles; one factorizes the denominator as $(2z+1)(2z-i)$, so that the poles are at $z_1 = -1/2$ and $z_2 = i/2$, i.e., $\mathcal{C}$ encloses z_1, $\mathcal{C}'$ encloses z_2 and $\mathcal{C}''$ encloses neither, from which we can already conclude that the integral over the later trajectory is zero. The integrals over the other paths are then:

$$\oint_{\mathcal{C}} \frac{\sin(z)/(2z-i)}{(2z+1)}dz = 2i\pi\frac{\sin(1/2)}{1+i} = (1+i)\pi\sin(1/2) , \tag{8a}$$

$$\oint_{\mathcal{C}'} \frac{\sin(z)/(2z+1)}{(2z-i)}dz = (i-1)\pi\sinh(1/2) . \tag{8b}$$

4. A limit z_l of a sequence (z_n) is a point such that, for all $\epsilon > 0$, there exists $\nu \in \mathbf{N}$ such that if $n \geq \nu$ then $|z_n - z_l| < \epsilon$. A point of accumulation z_0 of a sequence (z_n) is a point such that, for all $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $|z_N - z_0| < \epsilon$. It is clear that a limit is a stronger condition than a point of accumulation, since the latter requires that some points satisfy the criterion while the former requires that *all* points do. It is enough to take $N = \nu$ (the first point) to prove that a limit is a point of accumulation.

5. If z_1 and z_2 are both a limit for (z_n), then there exists N such that for any $\epsilon > 0$, $|z_1 - z_n| < \epsilon$ and $|z_2 - z_n| < \epsilon$ for all $n > N$. Now $|z_1 - z_2| \leq |z_1 - z_n| + |z_2 - z_n| \leq 2\epsilon$, i.e., $z_1 = z_2$. This shows that while a limit is a point of accumulation, the reverse is not true, since if there is a limit, it is unique, while there can be many points of accumulation (or none) for a series.