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Spontaneous coherence buildup in a polariton laser

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We study a toy model of two oscillators reaching equilibrium through coupling to a thermal bath to gain insights into the mechanism responsible for coherence buildup in an assembly of conserved bosons. We show how, in some conditions, coherence can appear spontaneously out of thermal states, i.e., without prior existence in the system. We then study the dynamics when particles have a finite lifetime and the overall mean number is maintained by pumping.

1 Introduction

In this paper we discuss how coherence arises in an assembly of bosons accumulating in a single state, when their energy dissipation is through scattering to lower energy states, as opposed to such bosons like photons or phonons where particles can be simply created or annihilated. To that effect we recourse to a toy model which reduces to the bare minimum many conceivable systems of greater complexity, but especially one of interest to the authors and to the audience of this conference: microcavity polaritons [1]. In such systems a so-called *polariton laser* effect is expected [2] where polaritons—bosons at low densities—gather in the system's ground state and upon radiative recombination emit coherent light, owing to their common quantum nature. As such it has much in common with the Bose Einstein Condensation (BEC) of atoms, and thus resembles more to an atom laser [3] than to a conventional laser. However it has also specificities of its own, namely polaritons have a short lifetime and their dispersion relation leads to a bottleneck of relaxation towards ground state for exchange of small momenta. Therefore, although based on macroscopic occupancy of a single state, the dynamics is crucial in this problem and makes it disputable that BEC is involved in microcavities. For the first thing, polaritons are two dimensional bosons and BEC does not appear in this dimensionality without a confining potential. For the second, because of the finite lifetime, an hypothetic condensate can be sustained only if the device is operated out of equilibrium. The polariton laser therefore sits between an atom laser and a conventional (photon) laser, the former operating at equilibrium based on BEC, the latter operating far from thermal equilibrium, based on a flux equilibrium.

Since dimensionality is not an issue because not the accomodation of a population in phase-space but dynamical effects are responsible for populating the ground state, we shall describe the system by a zero-dimensional two-oscillators model, one oscillator figuring the ground state, the other an excited state. In next section we discuss the notion of coherence in a BEC as well as various theoretical efforts devoted to its understanding; in section three we introduce a rate equation which displays the mechanism responsible for apparition of coherence; in section four we apply it to exhibit possibility of coherence buildup out of

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thermal states (without coherence previously existing in the system); in section five we study the dynamics of this buildup when particles have a finite lifetime, thus providing a simple model for a polariton laser.

2 Other works and definition of coherence

There have been various theoretical works related to growth of a condensate. Put aside Einstein's insight [5] based on statistical arguments, most authors favored a semi-classical Boltzmann equation [4], which however resulted in a majority of negative results at the exception of some works held in finite size system, dispensing from or postponing the thermodynamic limit. The semi-classical Boltzmann equation (SCBE) is unable to initiate the condensate and can only squeeze the distribution to temperature vanishingly higher than critical. However if the condensate is already present, it properly describes the redistribution of particles between it and its "vapor". A way out of this difficulty is the introduction of a "seed" which bypasses whatever actual mechanism is required. Therefore the growth has been divided in three stages, two kinetic stages where SCBE holds, separated by a *coherent stage* where it is invalid and where some other mechanism is responsible for nucleating the condensate. The coherent stage has been studied by Kagan *et al.* [7] who further split it into many substages, essentially one which first decays the amplitude fluctuations of the order parameter (which is in this case $\langle\psi(\mathbf{r})\rangle$ with ψ the wavefunction, see [8] for details). Then another stage decays phase fluctuations of the order parameter. Gardiner *et al.* covered at depth the issue of quantum kinetics in presence of a condensate, or in a regime where one should arise, in a series of seven papers entitled QKI to QKVII [9] where a hierarchy of quantum kinetic equations are derived, applied and simulated for the case of cold atoms. The simplest equation able to grow condensation without seed is, in their terminology, a Quantum Boltzmann Master Equation (QBME), i.e., an equation which structure essentially follows that of the usual SCBE but which is an equation for the probability distribution of the state's configurations rather than for its populations (which are mean values only). Namely, in a Fock basis $|n_0, n_1, \dots\rangle$ which describe a system with n_i particles in state i (in some basis), the QBME is an equation of motion for $p(n_0, n_1, \dots)$ the probability of the system to be found in this state. One recovers SCBE from QBME by neglecting fluctuations and correlations between states, i.e., if one assume such identities as $\langle n_i n_j\rangle = \langle n_i\rangle\langle n_j\rangle$. Recently, we studied with Y. G. Rubo [10] a master equation for the ground state alone, so that although it permits the exploration of quantum features of this state, like its phase survival, it is unable to grow coherence out of vacuum and as in the Boltzmann description, a seed has been used (see however [11] in this volume where this theory is extended to dispense from the seed). We shall use a QBME for the simplest scattering processes in a two-oscillators system in next section.

As for coherence itself, since we are dealing with a single mode, we expect, independently of the quantum state, strong first order coherent features, i.e., the emission will be in a narrow spectrum and in the ideal limit of noninteracting, infinite lifetime particles, the spectrum is actually a delta function $\delta(\hbar\omega - E_1)$. If one takes into account self-interaction, collision induced dephasing or finite lifetime, the spectrum is consequently homogeneously broadened [12]. We do not take much concern in this aspect which in this special case is largely independent of actual coherent quality of the state. For a single state, one refers more profitably to the higher coherence degree of the state, related to *counting statistics* of the state. If the bosonic annihilation operator for state i is a_i, the zero-delay second order coherence is defined [13] by

$$g_i^{(2)}(0) = \frac{\langle a_i^\dagger a_i^\dagger a_i a_i\rangle}{\langle a_i^\dagger a_i\rangle^2} = \frac{\sum_{n_1,n_2} n_i(n_i-1)P(n_1,n_2)}{\left[\sum_{n_1,n_2} n_i P(n_1,n_2)\right]^2} \tag{1}$$

where the average of middle expression is quantum average, and in the case $i = 1, 2$ of a two-oscillators system—that we study in this paper—for rightmost expression. This depends heavily on the quantum state. It ranges from 1 for a coherent state—with distribution $p_{\mathrm{coh}}(n) = e^{-\alpha^2}\alpha^n/\sqrt{n!}$ (α^2 the intensity)—to 2 for a thermal state—with geometric distribution $p_{\mathrm{th}}(n) = (1 - e^{\tau})e^{\tau n}$ (with $\tau \propto -1/T$, T the temperature). Physically, it accounts for the correlations of detection events. In the case of a coherent state, particles are independent and the probability to detect one does not affect subsequent probabilities of detection. The thermal state exhibits bunching in that sense that a detection increases probability of another detection. Another limiting case of interest is the Fock state, where $g^{(2)}(0) < 1$, corresponding to antibunching: detecting a particle in a well defined number state lowers probability of another detection. In term of moments, (1) reads $(\langle n^2\rangle - \langle n\rangle)/\langle n\rangle^2$, so that if $g^{(2)}(0) = 1$ the variance of particle number equals the particle number, which is characteristics of a gaussian. If $g^{(2)}(0) = 2$ the variance is higher by the square of the particle number. The relative mean squared deviation lowers with increasing particle numbers for gaussian-like statistics whereas it increases for thermal states. Physically this means a coherent state can be understood as a *single state* with stable nonzero intensity of emission (i.e., with low fluctuations about this given intensity), whereas a thermal state is essentially not emitting apart from occasional, greatly fluctuating, emission.

The exact distributions $p_{\rm coh}$ and $p_{\rm th}$ are limiting case fulfilled under realistic conditions where there is an infinity of oscillators, rather than just two. The main characteristics that we shall adopt to qualify a state as thermal or coherent in our special case will be much less stringent but still much more accurate than a mere "close to" criterion: we define a state as thermal or essentially thermal if its probability of occupancy is higher for the vacuum and decreasing with occupation number, i.e., $p_{\rm th}(n+1) < p_{\rm th}(n)$, whereas a state is qualified as coherent if its higher probability of emission is not centred about 0, i.e., if the distribution exhibits a peak at some nonzero, preferably high, value. In this way a Fock state would appear coherent when it is manifestly not. Such special cases however will not be met in the growth process and the distribution will remain between genuine thermal and coherent states. The intent of such conventions is merely to simplify the quest of criterion for coherence growth in our special case and not to loosen already existing definitions of general validity. States which arise in practise in the vicinity of critical conditions are flattened geometric distribution or stretched gaussians, reflecting a mixed state with some coherence and some thermal contamination. For such a state $g^{(2)}(0)$ will sit between 1 and 2. To summarise, we understand coherence as emission with number fluctuation locking. All this work therefore concentrates on $p_i(n)$ the probability distribution, or statistics, of state i, and especially of ground state, without considerations for the phase or any symmetry breaking which requires off-diagonal elements of the density matrix. In this way we concentrate on Kagan's first stage of decay of fluctuations in amplitude. Moreover, experimentalists currently measure nothing out of scope of $p(n)$ alone and to this date, the best experimental evidence for polariton condensation has been through the observation of decrease of $g^{(2)}(0)$ [14].

3 QBME for the two-oscillators model

Detailed investigations of the coherence buildup in bosonic systems have been actively pursued but the complexity of the proposed theories often hinder analytical understanding and heavy computations are required. Especially, Gardiner *et al.* applied to cold atoms technics from quantum optics, like stochastic or quantum state diffusion methods [9], which are intrinsically numerical. To gain physical insights into the mechanism at work in the growth of coherence, which dispense from first principle simulations, we study at depth the simplest case conceivable which captures all the essential physics, namely a two-oscillators bosonic system, where inter-particles interactions are neglected and with relaxation mechanisms that conserve particle numbers. The first approximation holds in the low density region which is satisfied when the condensate is forming out of the vacuum [10]. The second is an intrinsic property of polaritons which are conserved particles (in the sense that a particle—like an atom—is conserved, as opposed to a gauge boson, like a photon or phonon). This results in correlations of occupancy numbers of the different states, since if a polariton enters a state, another state has lost this polariton. Also the number of polaritons in the entire system fluctuates, but we shall see that the correlations implied by conservation of polaritons in their relaxation is at the heart of our mechanism. One can reconcile the conservation of particles with a fluctuation in their total number with an interpretation as a pulsed experiment, where a laser injects periodically in time a fluctuating number of particles in the system. Each relaxation by its own is for an exact and constant number of particles, while observed results are averaged over pulses and thus echo an overall fluctuating population.

States will be labelled 1 and 2. There is only one parameter to distinguish them which is the ratio ξ of the rate of transitions $w_{1\to2}$ and $w_{2\to1}$ between states:

$$\xi \equiv \frac{w_{2\to1}}{w_{1\to2}} \tag{2}$$

Those are constants and we shall assume $\xi > 1$ which identifies state 1 as the ground state (i.e., state of lower energy), since from elementary statistics:

$$\frac{w_{2\to1}}{w_{1\to2}} = e^{(E_2-E_1)/kT} \tag{3}$$

with E_i the energy of state i (by definition of ground state $E_2 > E_1$) and T is the temperature of the system once it has reached equilibrium.

For such a two-oscillators system, assuming the simplest scattering channel of diffusion by emission or absorption of a phonon, the QBME reads:

$$\begin{aligned}\dot{p}(n,m) = {}& (n+1)m[w_{1\to2}p(n+1,m-1) - w_{2\to1}p(n,m)] \\ & + n(m+1)[w_{2\to1}p(n-1,m+1) - w_{1\to2}p(n,m)]\end{aligned} \tag{4}$$

where $p(n,m)$ is the joint probability distribution to have n particles in state 1 and m in 2. This equation has a very clear physical significance and one hardly needs to refer to a more general QBME, rigorously derived from a microscopic hamiltonian. For instance the first term of rhs expresses that one can increase the probability to have (n,m) particles in states $(1,2)$ through the process where starting from $(n+1,m-1)$ configuration, one reaches (n,m) by transfer of one particle from state 1 to the other state. This is proportional to $n+1$, the number of particles in state 1 and is stimulated by $m-1$ the number of particles in state 2 to which we add one for spontaneous emission, whence the factor $(n+1)m$. We repeat that $w_{1\to2}$ and $w_{2\to1}$ are constants and should not be confused with the *bosonic transition rate* defined as $w_{1\to2}(1+m)$ and $w_{2\to1}(1+n)$ to account in a transparent way for stimulation. Our present discussion will be clarified by expliciting it.

We will soon undertake to solve exactly this equation but to delineate its quality in explaining how coherence arises in the system we first show that if we make the approximation to neglect correlations between the two states, i.e., if we assume the factorisation

$$p(n,m) = p_1(n)p_2(m) \tag{5}$$

then the system at equilibrium will never display any coherence, i.e., in accord with our previous discussion, both states will be in a thermal state no matter the initial conditions, the transition rates or any other parameters describing the system. Indeed put (5) into (4) and sum over m to obtain:

$$\begin{aligned}\dot{p}_1(n) = {}& p_1(n+1)w_{1\to2}(n+1)(\langle m\rangle+1) \\ & - p_1(n)\big(w_{2\to1}(n+1)\langle m\rangle + w_{1\to2}n(\langle m\rangle+1)\big) \\ & + p_1(n-1)w_{2\to1}n\langle m\rangle\end{aligned} \tag{6}$$

with $\langle m\rangle \equiv \sum_m m p_2(m)$ the average number of bosons in state 2. At equilibrium the detailed balance of these two states gives the solution

$$p_1(n+1) = \frac{\langle m\rangle}{\langle m\rangle+1}\frac{w_{2\to1}}{w_{1\to2}}p_1(n) \tag{7}$$

The same procedure for state 2 yields likewise

$$p_2(m+1) = \frac{\langle n\rangle}{\langle n\rangle + 1}\frac{w_{1\to 2}}{w_{2\to 1}}p_2(m) \tag{8}$$

With the notational shortcuts

$$\theta \equiv \frac{\langle n\rangle}{\langle n\rangle + 1}, \qquad \nu \equiv \frac{\langle m\rangle}{\langle m\rangle + 1} \tag{9}$$

equations (7) and (8) read after normalisation $p_1(n) = (1-\nu\xi)(\nu\xi)^n$ and $p_2(m) = (1-\theta/\xi)(\theta/\xi)^m$, so that $\langle n\rangle \equiv \sum np_1(n) = \nu\xi/(1-\nu\xi)$ which inserted back into (9) yields

$$\xi = \frac{\theta}{\nu} \tag{10}$$

or, written back in terms of occupancy numbers and transition rates:

$$\frac{w_{2\to 1}}{w_{1\to 2}} = \frac{\langle n\rangle}{\langle n\rangle + 1}\frac{\langle m\rangle + 1}{\langle m\rangle} \tag{11}$$

which give in eq. (7), (8):

$$p_1(n+1) = \frac{\langle n\rangle}{\langle n\rangle + 1}p_1(n) \quad \text{and} \quad p_2(m+1) = \frac{\langle m\rangle}{\langle m\rangle + 1}p_2(m) \tag{12}$$

achieving the proof that both states are (exact) thermal states under the hypothesis (5) that we will now relax. This will give rise to a likewise regime where both states are thermal states, but also to another regime where the excited state (state 2) is still in a thermal state, but the ground state (state 1) is non-thermal (and in some limit, has the statistics of a coherent state). This is possible if one takes into account correlations between states. In our case these correlations come from the conservation of particle number, so that the knowledge of particle number in one state determines the number in other state. In fact observe how the QBME connects elements of $p(n, m)$ which lie on antidiagonals of the plane (n, m). One such antidiagonal obeys equation

$$n + m = N \tag{13}$$

where N is a constant, namely, the distance of the antidiagonal to the origin from the geometrical point of view, and the number of particles from the physical point of view. One such antidiagonal is sketched on fig. 1. The equation can be readily solved if only one antidiagonal is concerned, i.e., if there are exactly N particles in the system. In case where the particle number in the system is only known probabilistically, one can still decouple the equation onto its antidiagonal projections, solve for them individually and add up afterwards weighting each antidiagonal with the probability to have the corresponding particle number. We hence focus on such one antidiagonal N which *conditional* probability distribution is given by $d(n|N) \equiv p(n, N-n)$, with equation of motion given by (4) as:

$$\begin{aligned}\dot{d}(n|N) = (n+1)(N-n)[w_{1\to 2}d(n+1|N) - w_{2\to 1}d(n|N)] \\ +n(N-n+1)[w_{2\to 1}d(n-1|N) - w_{1\to 2}d(n|N)]\end{aligned} \tag{14}$$

The equation is well behaved on regard of its domain of definition $0 \le n \le N$ since it secures that $d(n|N) = 0$ for $n > N$. This also ensures unicity of solution despite the recurrence solution being of order 2, for $d(1|N)$ is determined uniquely by $d(0|N)$, itself determined by normalisation.

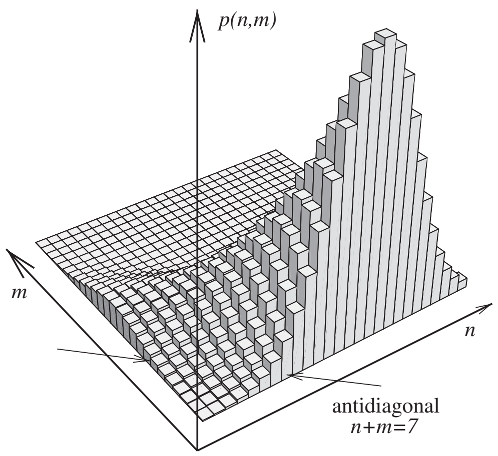

Fig. 1 $p(n, m)$ steady state solution in case where the distribution function for the number of particles in the entire system $P(N)$ is a gaussian of mean (and variance) 15 and $\xi = 1.2$. One "antidiagonal", $n + m = 7$, is shown for illustration. The projection on n-axis displays a coherent state whereas the projection on m-axis displays a thermal state.

The stationary solution is obtained in this way (or from detailed balance):

$$d(n+1|N) = \frac{w_{2\to 1}}{w_{1\to 2}} d(n|N), \tag{15}$$

with solution

$$d(n|N) = d(0|N) \left(\frac{w_{2\to 1}}{w_{1\to 2}}\right)^n \tag{16}$$

where $d(0|N)$ is defined for normalisation as

$$d(0|N) = \frac{\xi - 1}{\xi^{N+1} - 1}, \qquad \xi \equiv \frac{w_{2\to 1}}{w_{1\to 2}} \tag{17}$$

Technically the solving procedure resembles much that already encountered to solve the equation under assumption (5), only we are now paying full account for correlations of two states, which turned detailed balancing (7) and (8) into one of an altogether different type (16). This gives by weighting (16) the solution to the QBME:

$$p(n, m) = \frac{\xi - 1}{\xi^{n+m+1} - 1} \xi^n P(n+m) \tag{18}$$

with $P(N) \equiv \sum_{n+m=N} p(n, m)$ the distribution of total particle number, i.e., probability to have N particles in the *entire* system. $P(N)$ is time independent since the microscopic mechanism involved conserves particle number for any transition (one can also check that $\dot{P}(N) = 0$). This allows to derive the statistics of separate states:

$$\begin{aligned} p_1(n) &= \xi^n \sum_{N=n}^{\infty} \frac{\xi - 1}{\xi^{N+1} - 1} P(N) \\ p_2(n) &= \xi^{-n} \sum_{N=n}^{\infty} \frac{\xi - 1}{\xi^{N+1} - 1} \xi^N P(N) \end{aligned} \tag{19}$$

Observe how the n dependence of the sum index prevents trivial relationship between p_1 and p_2 of the kind $p_1(n) = p_2(N - n)$. Also the asymmetry between ground and excited state is patent from (18). It is this feature which allows to have two states with drastic different characteristics, typically a thermal and a coherent state. Indeed, p_1 (resp. p_2) is the product of a sum with an exponentially diverging (resp. converging to zero) function of ξ. In both cases, the sum of positive terms is a decreasing function of n, so that clearly no coherence can ever survive in excited state which fate is always thermal equilibrium, or at least, in accord with our definitions,

$$p_2(n) > p_2(n + n_0) \quad \text{for all } n, n_0 \text{ in } \mathbf{N} \tag{20}$$

For p_1 however, ξ^n diverges with n which leaves open the possibility of a peak not centred about zero in this distribution, while it can still be a decreasing function if the sum converges faster still. It is to $P(N)$ to settle this issue, which as a constant of motion is completely determined by the initial condition. The solution for the case where $P(N)$ is a gaussian of mean (and average) 15 is displayed on fig. 1. $p(n, m)$ is in this case manifestly not of the type $p_1(n)p_2(m)$ and there is always coherence in the system. In next section we investigate the more interesting situation where coherence is not existing a-priori in the system.

4 Growth at equilibrium

By growth at equilibrium we mean that, still in the approximation of infinite lifetime, coherence can arise when one lowers temperature, i.e., increases ξ, in a system where initially all states are thermal states. In this case the initial condition for the system is the thermal equilibrium

$$p(n, m) = (1 - \theta)(1 - \nu)\theta^n \nu^m \tag{21}$$

where θ, ν are the thermal parameters for ground and excited states respectively. They link to $\langle n \rangle$, the mean number of particles in ground state, through $\langle n \rangle = \theta/(1 - \theta)$, or, the other way around, $\theta = \langle n \rangle/(1 + \langle n \rangle)$. Similar relations hold for ν. This is one possible steady solution of (4) and we discuss how it arises from (18) below. For the time being we stress again that a thermal state is essentially empty, as attests its higher probability which is for zero particle. Once in a while, thermal kicks transfer in the state one or many particles, which however do not stay for long before the state is emptied again or replaced by other, unrelated particles. This accounts for the chaotic, or incoherent, properties of such a state. This essentially empty but greatly fluctuating statistics brings no conceptual problem for little populations, but one might enquire whether it is conceivable to have a thermal distribution with high mean number. This is possible for a single state but not for the system at once. A macroscopic population can distribute itself in a vast collection of states so that each has thermal statistics, constantly exchanging particles with other states and displaying great fluctuations, but as expected from physical grounds, the whole system does not fluctuate greatly in its number of particles. Therefore we expect $P(N)$, the distribution of particles in the *entire system*, to be peaked about a nonzero value, typically to be a gaussian of mean and variance equal to N. Reminding our previous definitions, this however does not qualify the system as a coherent emitter, since the statistics must refer to a *single state*, not to a vast assembly of differing emitters. Thus, not surprisingly, coherence arises when a *single* quantum mode models or copy features of a macroscopic system, typically its population distribution. The two-oscillators system which is a rather coarse approximation to

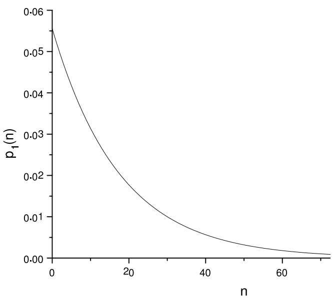

Fig. 2 Ground state distribution $p_1(n)$ for $\xi = \theta/\nu \approx 1.26$, with 3 particles in excited state and 17 in ground state, both in thermal equilibrium. $g^{(2)}(0) = 2$.

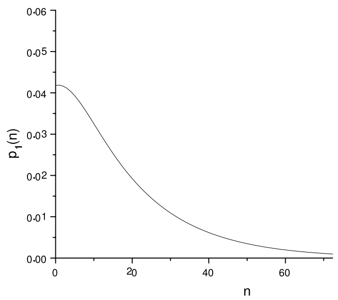

Fig. 3 Same with ξ raised to 1.5. Distribution is non-thermal, especially $p_1(1) > p_1(0)$, though the distribution is then decreasing. $g^{(2)}(0) \approx 1.89$.

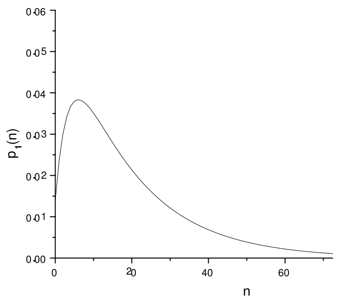

Fig. 4 Same with ξ increased to 10^5. Distribution is distinctively non-thermal, yet far from coherent in the two-oscillators model. $g^{(2)}(0) \approx 1.74$.

a macroscopic system will however display very clearly this mechanism. In the limit where $\xi \gg 1$ it is already visible from (19) that $p_1(n) \approx P(n)$, so that the statistics of the entire system indeed serves as a blueprint for the ground state (and it alone, excited states being always decreasing as already shown). At equilibrium, with two thermal states, the distribution for the whole system reads:

$$P(N) = \sum_{n+m=N} p(n,m) = (1-\theta)(1-\nu)\frac{\theta^{N+1} - \nu^{N+1}}{\theta - \nu} \tag{22}$$

This exhibits a peak at a nonzero value provided that

$$\nu + \theta > 1 \tag{23}$$

This criterion refers to a first necessary condition: there must be enough particles in the system. The less particles available so that (23) is fulfilled, is two. This minimum required to grow coherence fits nicely with the Bose-Einstein condensation picture (one needs at least two bosons to condense). It is not a necessary condition, though; also the dynamical aspect is important as shown by the key role of ξ. Indeed if the system is steady in configuration (21), ξ is not a free parameter but is related to θ and ν by:

$$\xi = \frac{\theta}{\nu} \tag{24}$$

and in this case the distribution of ground state

$$p_1(n) = (1-\theta)(1-\nu)\xi^n \sum_{N=n}^{\infty} \frac{\theta^{N+1} - \nu^{N+1}}{\xi^{N+1} - 1}\frac{\xi - 1}{\theta - \nu} \tag{25}$$

reduces by straightforward algebra to

$$p_1(n) = (1-\theta)\theta^n \tag{26}$$

i.e., as should be for consistency, the ground state is in a thermal state, independently of the value of θ (i.e., no matter the number of particles in ground state). This can come as a surprise, but it must be born in mind that this two-oscillators model is an extreme simplification which cannot dispense from some pathological features, namely, the faculty to sustain a thermal macroscopic population, an ability that we understand easily since the ground state accounts for half of the system! It is expected that with increasing number of states, dimensionality will forbid such an artifact. Also the shape of $P(N)$ hardly resembles a gaussian (see fig. (4)) but already in this limiting case it is able to display a peak at a nonzero value provided there are enough particles. With increasing number of states, the central limit theorem will turn this distribution into an actual gaussian. Once again, $P(N)$ is time independent because the relaxation mechanism conserves particle number, which results in correlations between the two states. By increasing ξ to ξ', one might search new values of θ, ν, say θ', ν', so that $\theta/(1-\theta)+\nu/(1-\nu) = \theta'/(1-\theta')+\nu'/(1-\nu')$ (conservation of particle number) and $\xi' = \theta'/\nu'$. This is possible if one allows $P(N)$ to change, in which case the two new states are also thermal states. If $P(N)$ is constrained by correlations induced by strict conservation of particle numbers, so that the uncertainty is not shifted as the system evolves, then (24) breaks down and this allows (18) to grow a coherent state in ground state. This process is illustrated on figures 2–4, starting frow thermal equilibrium and lowering temperatures (increasing ξ). In the two-oscillators model, coherence grown out of thermal states cannot come much closer to a gaussian than illustrated on fig. 4.

5 Growth out of equilibrium

The previous case holds in an equilibrium picture and for that matter refers to coherence buildup in systems like cold atoms BEC. To address the polariton laser case, it is necessary to extend the two-oscillators model with the additional complications of finite lifetime τ of particles in state 1, with a balance in the total population provided by a pump which inject particles in state 2 at a rate Γ. (We will not crucially need finite lifetime in excited state and thus neglect it, which is good approximation in a typical microcavity where the radiative lifetime drops by a factor of ten to an hundred in the photon-like part of the dispersion.) Although the QBME can be readily extended phenomenologically to take these into account,

$$\begin{aligned}\dot{p}(n,m) &= (n+1)m[w_{1\to2}p(n+1,m-1) - w_{2\to1}p(n,m)] \\ &\quad + n(m+1)[w_{2\to1}p(n-1,m+1) - w_{1\to2}p(n,m)] \\ &\quad + \frac{1}{\tau}(n+1)p(n+1,m) - \frac{1}{\tau}np(n,m) \\ &\quad + \Gamma p(n,m-1) - \Gamma p(n,m)\end{aligned} \tag{27}$$

the couplings between different particle numbers forbid to solve this new equation along the same analytical lines as previously. However in this out-of-equilibrium regime, the excited state is not as important as in the equilibrium case where it must be thermal and which configuration is of utmost consequence on the ground state. Thus we can dispense from the actual distribution of excited state and content with the mean $\langle m\rangle_n$ obtained from $\sum_m mp(n,m) = \langle m\rangle_n p_1(n)$. In this case (27) can be decoupled to give an equation for $p_1(n)$ alone, and in the "dynamical" steady state, the detailed balance reads:

$$p_1(n+1) = \frac{w_{2\to1}\langle m\rangle_n}{w_{1\to2}(\langle m\rangle_{n+1}+1)+1/\tau}p_1(n) \tag{28}$$

Up to this point it is still exact, only we do not take further interest in the excited state's statistics. Also in this out-of-equilibrium case we grant the conservation of particle number as the origin of correlations between the two states, but because of lifetime and pumping, it can now be secured only in the mean, leading us to the following approximation for $\langle m\rangle_n$:

$$\langle m\rangle_n = N - n \tag{29}$$

The pump, which has quantitatively disappeared from the formula, is implicitly taken into account through this assumption, since even though particles have a finite lifetime, their number is constant on average.

When the population has stabilised in the ground state by equilibrium of radiative lifetime and pumping, it is found in a coherent state if $N > N_c$ with N_c the critical population defined by:

$$N_c = \frac{1}{\tau(w_{2\to1} - w_{1\to2})} \tag{30}$$

and obtained from (28) and (29) with the requirement that $p_1(1) > p_1(0)$. If this population is exceeded, coherence builds up in the system along with the population, which stabilises at an average given by the maximum of the gaussian-like distribution:

$$\langle n \rangle = n_{\max} = N - N_c \tag{31}$$

obtained from $p_1(n) = p_1(n+1)$; so that effectively if $N < N_c$ there is no such gaussian and coherence remains low with a thermal-like state which maximum is for zero occupancy. If $N > N_c$ the state is a gaussian which mean increases with increasing departure of population from the critical population. Thus, the more the particles, the less the particle number fluctuations of the state, the best its coherence.

On fig. 5 is displayed the numerical solution of eq. (27) for $p_1(n, t)$, where parameters (see legend) have been chosen so that N exceeds (30) and therefore grow some coherence from an initially empty ground state (cf. fig. 5-a). The coherence is maintained for infinite times and the statistics for the ground state occupancy tends towards a gaussian-like function neatly peaked about a high value (cf. fig. 5-b). We define a *coherence degree* equal to $2 - g^{(2)}(0)$, so chosen to be 0 for a genuine thermal state and 1 for a genuine coherent state. In the case where (30) is exceeded, the coherence degree of ground state quickly reaches unity (fig. 5-c).

On fig. 6 is displayed the counterpart situation where parameters (see legend) result in a sub-critical population so that the steady state is thermal, as shown on fig. 6-b. The dynamics of $p(n)$, starting from vacuum, is merely to grow this thermal state (cf. fig. 6-a) and the coherence degree remains low (cf. fig. 6-c).

In the limit of infinite lifetime, one recovers the result of the previous section in the case where $P(N)$ is a Kronecker delta. The dynamics of coherence buildup in a realistic microcavity is complete when one extends (28) such that instead of a single excited state one accounts for all excited states of the system with suitable correlations with ground state (with possibility to neglect correlations between excited states as a simplification), and by using parameters (lifetime and transition rates) of a realistic cavity as computed by SCBE. Such results are the topic of a future publication.

6 Conclusions

We derived from the application of a Quantum Boltzmann Master Equation (QBME) to a simple two-oscillators model—that we solved analytically, eq. (18)—some insights into the physics at work in the process of coherence buildup in an assembly of conserved bosons. We identify in this case correlations between states as the responsible mechanism, in sharp contrast with lasers where nonlinearities provided by a resonator are locking the fluctuations. In the case studied, correlations are caused by strict particle number conservation: the number of particle in one state determines the number in the other state. We showed that enough bosons (at least two) and efficient rate of transitions between states is needed. Coherence then arises when the distribution function for the number of particles in ground state behaves like the distribution for the entire system. Even in the two-oscillators model, it is thus possible to grow some coherence out of thermal states. The coherence is more efficient the higher the correlations. Finally, we addressed the kinetics of coherence buildup when particle have finite lifetime. Depending on whether the system has or has not time to populate the ground state (cf. eq. (30)), this one grows some coherence or remains thermal with low population.

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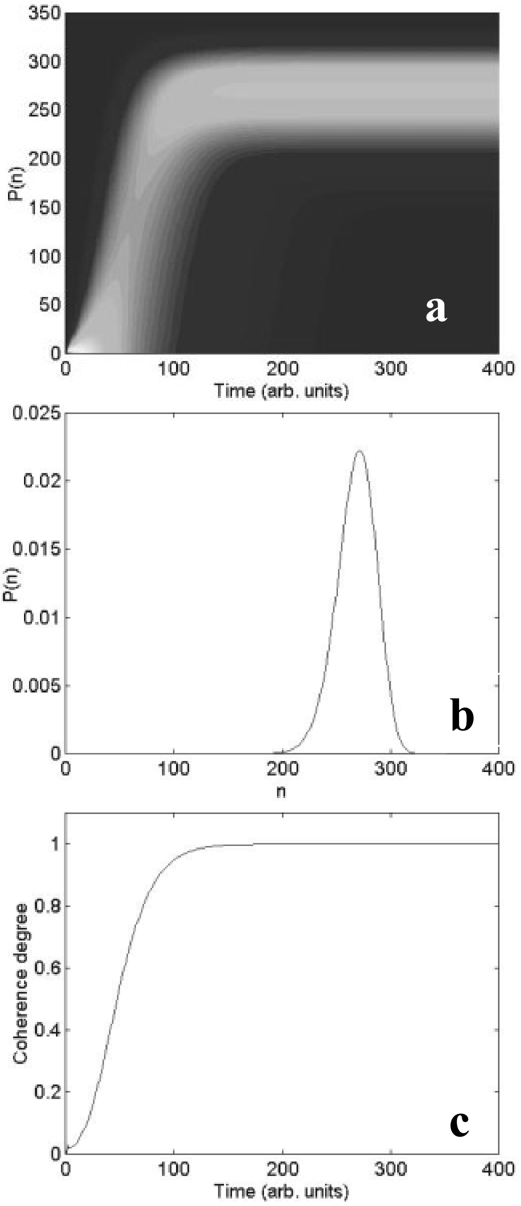

Fig. 5 System configuration suitable for coherence buildup: $N = 350$, $w_{2\to1} = 10^{-5}$ (arb. units), $w_{1\to2} = 0.75\times10^{-5}$ (arb. units) and $1/\tau = 20\times10^{-5}$ (arb. units). All units have the same dimension of an inverse time. (a) is a density plot for the time evolution of $p(n)$, starting from vacuum it quickly evolves towards a coherent state. (b) is the projection of $p(n)$ in the steady state. (c) is the time evolution of the coherence degree $2 - g^{(2)}(0)$: full coherence is quickly attained.

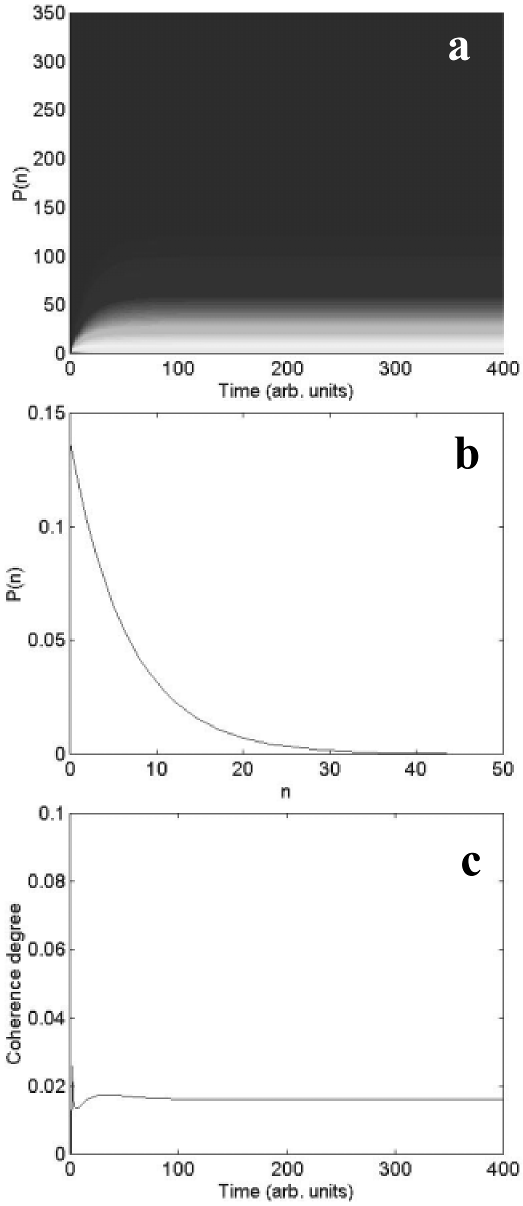

Fig. 6 System configuration unable to develop coherence. Parameters are the same as for fig. 5 except $w_{2\to1} = .95 \times 10^{-5}$ (arb. units) corresponding to a higher temperature. (a) Starting from vacuum the ground state steadies in a thermal state for which a projection (b) is shown. (c) The coherence degree remains low.

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