MÉTODOS MATEMÁTICOS II
Answers to the partial exam.

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I. COMPLEX CALCULUS

- i^n takes the values $i, -1, -i, 1$ and then cycles through this again for $n = 1, 2, 3, 4$ and counting. We therefore have to figure out with which value $n = 12345$ is associated to. We write it $i^{12345} = i \times i^{12344}$ and since $12344 = 12000 + 300 + 44 = 2 \times (600 + 150 + 22) = 2 \times 772$, we find $i^{12344} = i^{2\times 772} = (i^2)^{772} = (-1)^{772}$. Since 772 is even, the sign is squared, and the result is:

$$i^{12345} = i\,. \tag{1}$$

- We first compute $z = \ln(i)$, the complex number z such that $\exp(z) = i$, that is, from the polar representation, $z = i\pi/2$. The result now follows from $\ln(i\pi/2)$, i.e., on the principal branch:

$$\ln(\ln(i)) = i\pi/2 + \ln(\pi/2)\,. \tag{2}$$

  To sketch the result, we will use $\ln(i) + \ln(\pi/2) =\approx 1.57i + b$ with $0 < b < 1$ from the properties of the logarithm (with $\ln(1) = 0$ and $\ln(e) = 1$ with $e \approx 2.71$.

- We start by the bottom of the continued fraction:

  1. $1/(1 + i) = (1 - i)/|1 + i|^2 = (1 - i)/2$ so $i + 1/(1 + i) = (1 + i)/2$.
  2. $1/[(1 + i)/2] = (1 - i)$ using the previous result so the second step from bottom is simply 1.
  3. $i + 1/1 = 1 + i$ and we are back at the step at which we started.

  The continued fraction will thus oscillate (as is typical of complex numbers) between the values $(1 + i)/2$, 1 and $1 + i$. As we have five stages of fractions, the result is:

$$i + \cfrac{1}{i + \cfrac{1}{i + \cfrac{1}{i + \cfrac{1}{i + \cfrac{1}{1+i}}}}} = 1\,. \tag{3}$$

To plot the inverse, the conjugate and the opposite of each case, we remind the geometric properties of these:

1. The conjugate is the image wrt the x axis.
2. The opposite is the image wrt the origin.
3. The inverse is the conjugate with inverse module.

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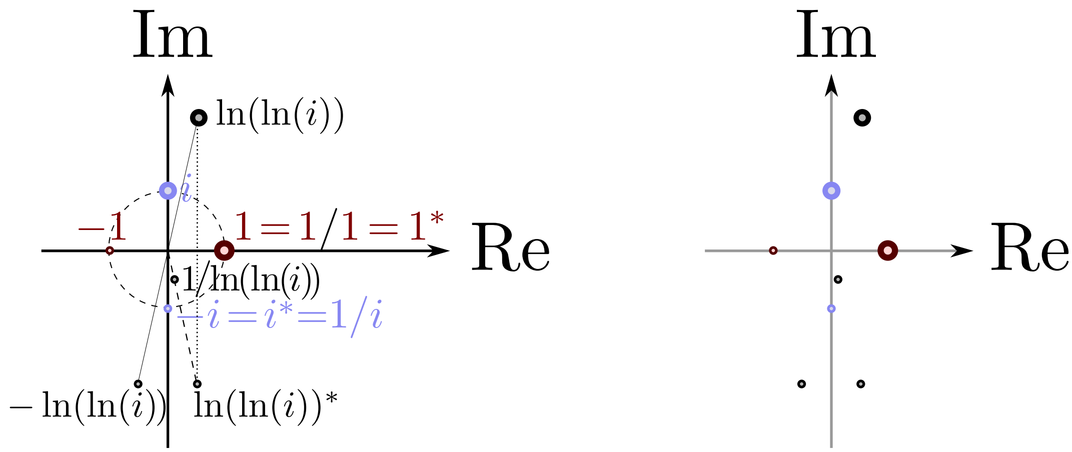

FIG. 1: Graphical representation of the solutions to exercise one, with (left) and without (right) labels.

II. MATHEMATICAL REASONING

If $|f| > |g|$ everywhere, then $|f| > 0$ everywhere since $|g| \geq 0$. Therefore f is never zero and, dividing, we have $h = g/f$ an entire function (everywhere holomorphic with derivative $(g'f - fg')/f^2$). Now since $|h| = |g/f| = |g|/|f|$ is bounded (by 1), h itself is, by Liouville's theorem, constant. Calling α this constant, $h = g/f = \alpha$.

To prove the stronger result with an inequality on the given particular cas, we invoke Cauchy's integral formula for the derivative:

$$f'(z) = \frac{1}{2i\pi} \oint \frac{f(w)}{(z-w)^2}\, dw \tag{4}$$

that, for any $z \in \mathbf{C}$, shows that $|f'(z)| \leq \frac{1}{2\pi} \times \mathrm{Max}(\frac{|f(z+R\exp(i\theta))|}{R^2}) \times 2\pi R$ (we have used as an integration contour a circle centered on z and of radius $R > 0$). Now since we have assumed $|f(z + Re^{i\theta})| \leq |c||z + Re^{i\theta}|$, we have, from the triangle inequality, $\mathrm{Max}(|f(z + Re^{i\theta})|) \leq |c|(|z| + R)$ and, putting everything together:

$$|f'(z)| \leq |c| \frac{|z| + R}{R} \tag{5}$$

with R arbitrary, that is, $|f'(z)| \leq |c|$ by taking the limit $R \to \infty$. We have just proved that f', itself an entire function, is everywhere bounded, that is, from Liouville's theorem again, f' is constant, and therefore, integrating, $f(z) = az + b$ with $a \neq 0$. At $z = 0$ the inequality reads $|b| \leq 0$, i.e., $b = 0$, thus $f(z) = \alpha g(z)$ with $\alpha = a/c$.

III. CONFORMAL MAPPING

The implicit equation reads:

$$\frac{z-(-1)}{z-1}\frac{0-1}{0-(-1)} = \frac{w-(-1)}{w-1}\frac{i-1}{i-(-1)}\,,$$

i.e., $-\frac{z+1}{z-1} = \frac{w+1}{w-1}\frac{(i-1)^2}{2}$, or, by further evaluation $-\frac{z+1}{z-1} = i\frac{w+1}{w-1}$, which we have to solve for w. Collecting such terms, we get $w(i + \frac{z+1}{z-1}) = -i + \frac{z+1}{z-1}$ and then $w = \frac{-i(z-1)+z+1}{i(z-1)+z+1}$, so that $w = \frac{z(1-i)+1+i}{z(1+i)+1-i}$. Dividing every term by $1-i$, since $(1+i)/(1-i) = i$, we find:

$$w = \frac{z+i}{zi+1}\,.$$

By direct computation, we find:

1. $-2 \to -(4+3i)/5$, which is on the unit circle (modulus is 1), on the third quadrant.

2. $-1/2 \to (-4+3i)/5$, also on the unit circle but now on the second quadrant.

3. $-i \to 0$.

Given that the Möbius transform describes inversion of the plane, and transforms lines into circles (possibly of infinite radius), with the above punctual transformations we can assert that:

1. The segment $-1 \leq x \leq 1$ is mapped to the half unit circle in the upper part of the plane.

2. The line $x < -1$ is mapped to the unit circle in the third quadrant of the plane.

3. The line $1 < x$ is mapped to the unit circle in the fourth quadrant of the plane.

Given that $-i$ is mapped to the origin, inside the unit circle, we conclude the half-complex plane $\mathrm{Im}(z) < 0$ is mapped to the open ball $|w| < 1$.

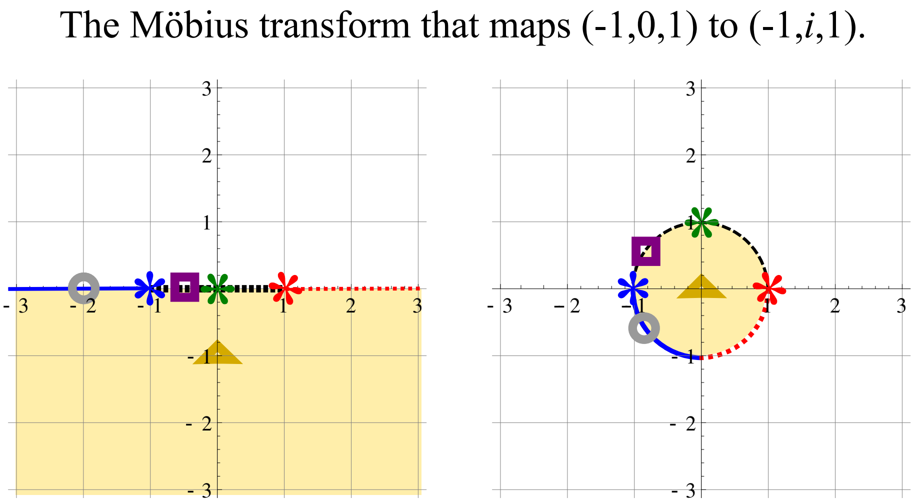

FIG. 2: Mapping of the points and regions given in the text through the conformal transform $\frac{z(1-i)+1+i}{z(1+i)+1-i}$.

IV. COMPLEX DIFFERENTIATION

We will use Cauchy-Riemann's condition, that we remind here. For $f(x+iy) = u(x,y) + iv(x,y)$, f is complex-differentiable iff

$$\partial_x u = \partial_y v \,, \tag{6}$$
$$\partial_y u = -\partial_x v \,. \tag{7}$$

We merely have to compute:

1. $u = x/y$ and $v = 0$ (the function is real), so that the CR conditions are satisfied iff $\partial_x(x/y) = 1/y = 0$ and $\partial_y(x/y) = -x/y^2 = 0$ which is never the case: the function is *nowhere* differentiable

2. $u = x^2y^3$ and $v = x^3y^2$, so that the CR conditions are satisfied iff $\partial_x(x^2y^3) = 2xy^3 = 2x^3y = \partial_y(x^3y^2)$ and $\partial_x(x^3y^2) = 3x^2y^2 = -3x^2y^2 = -\partial_y(x^2y^3)$, that is, iff $y^2 = x^2$ and $x^2y^2 = 0$. The first condition is the two diagonals $y = \pm x$ and the second the two axes $x = 0$ and $y = 0$. The intersection of these is the origin, which is therefore the only point where the function is complex-differentiable.

3. $u = x + y^2$ and $v = -2xy$, so that the CR conditions are satisfied iff $\partial_x(x+y^2) = 1 = -2x = \partial_y(-2xy)$ and $\partial_x(-2xy) = -2y = -2y = -\partial_y(x+y^2)$. The latter is always satisfied, so that the function is complex-differentiable on the line $x = -1/2$, where its derivative is $f' = \partial_x(u+iv) = 1 - 2iy$.

4. $f(x+iy) = \cos(i(x+iy))$ is an elementary function of z, namely, $f(z) = \cos(iz) = \cosh(z)$ that is everywhere differentiable (it is entire) with derivative $f' = \sinh(z)$.

5. The function is directly expressed in terms of holomorphic functions, so that, except at $z = 0$ where it is not defined, it is everywhere differentiable with derivative $f' = e^z(z-1)/z^2$.

V. COMPLEX INTEGRATION

- The function z^2+1 is holomorphic on a simply connected domain, therefore it is zero regardless of the path.
- Written as $\oint_{\mathcal{C}_{1,2}} \frac{dz}{(z-1)(z+1)}$, we see that the function has two poles, in ± 1. We use Cauchy's integral formula to 0th order to evaluate the contour integral on a circle centered on each pole, and find: $\oint_{|z-1|=1} \frac{dz}{(z-1)(z+1)} = 2i\pi \frac{1}{z+1}\big|_{z=1} = i\pi$ and $\oint_{|z-(-1)|=1} \frac{dz}{(z-1)(z+1)} = 2i\pi \frac{1}{z-1}\big|_{z=-1} = -i\pi$. The only difference between the two paths is that one changes the sense in the contour of integration around 1, therefore:

$$\oint_{\mathcal{C}_1} \frac{dz}{(z-1)(z+1)} = i\pi + (-i\pi) = 0 \quad \text{while} \quad \oint_{\mathcal{C}_2} \frac{dz}{(z-1)(z+1)} = -i\pi + (-i\pi) = -2i\pi.$$

- The function only has a pole in 1, where the contours on both paths go in opposite directions. Therefore, the results will be of opposite sign. We use Cauchy's integral formula to first order to compute $\oint_{\mathcal{C}_1} \frac{\cosh(z)}{(z-1)^2}\, dz = 2i\pi[\cosh(z)]'\big|_{z=1} = 2i\pi \sinh(1)$.
- The pole is at $i\pi/4$ which is within the unit circle, therefore, $\oint_{\mathcal{C}_3} \frac{e^z}{z-i\pi/4}\, dz = 2i\pi e^{i\pi/4} = -2\pi e^{-i\pi/4}$.
- This integral could be painful to compute explicitly (x is not holomorphic therefore we cannot use Cauchy integration). We can use the complex Green theorem, however, which, for $f = z^*$, gives $\oint_{\mathcal{C}} z^*\, dz = 2iA$ with A the area enclosed by $\mathcal{C}$. Now $\oint z\, dz + \oint z^*\, dz = 2\oint x\, dz$ with the first integral zero, and therefore:

$$\oint_{\mathcal{C}_4} \frac{x}{i}\, dz = 1 + \frac{\pi}{2} \approx 2.57 \tag{8}$$

  by summing the area of the two triangles (together forming a square of unit side) and half a circle of radius $r=1$, i.e., $\pi r^2/2$.