Mathematical Methods II
Handout 23. Residue theory.

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Residue theory is the culmination of complex integration, bringing together Cauchy's integral formula and Laurent Series. Integrating both sides of the Laurent expansion around z_0:

$$\oint f(z)\,dz = \sum_{n=-\infty}^{\infty} c_n \oint (z-z_0)^n\,dz\,, \tag{1}$$

and remembering that $\oint (z-z_0)^n\,dz = 2i\pi\delta_{n,-1}$, we find:

$$\oint f(z)\,dz = 2i\pi c_{-1} \tag{2}$$

The coefficient of the negative power of order one in the Laurent expansion is called the *Residue*. We write: $\mathrm{Res}_{z=z_0} f(z) = c_{-1}$. To calculate a residue at a pole, we need not produce a whole Laurent series, but, more economically, we can derive formulas to compute residues:

Simple poles:

$$\mathrm{Res}_{z=z_0} f(z) = \lim_{z\to z_0} (z-z_0)f(z)\,, \tag{3}$$

$$\mathrm{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}\,, \tag{4}$$

where, in the latter case, $f(z_0) \neq 0$ and z_0 is a simple zero of q.

The generalization to higher poles provides the formula:

$$\mathrm{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right] \right\}\,. \tag{5}$$

In particular, for a pole of order 2:

$$\mathrm{Res}_{z=z_0} f(z) = \lim_{z\to z_0} \{[(z-z_0)^2 f(z)']\,. \tag{6}$$

The so-called "*Residue Theorem*" extends this technique to the case of several singularities: if f is holomorphic inside a simple closed path $\mathcal{C}$ and on $\mathcal{C}$, except for finitely many points $z_1, \ldots, z_n$, inside $\mathcal{C}$, then the integral of $\mathcal{C}$ in the trigonometric sense is $2i\pi$ times the sum of the residues at the poles:

$$\oint_{\mathcal{C}} f(z)\,dz = 2i\pi \sum_{k=1}^{n} \mathrm{Res}_{z=z_k} f(z)\,, \tag{7}$$

The method of Residue is useful to compute integrals not only in the complex planes but also of the real variable. For instance, rational functions of $\cos(\theta)$ and $\sin(\theta)$ in polar coordinates are easily obtained through the substitution:

$$\cos(\theta) = \frac{1}{2}\left(z + \frac{1}{z}\right)\,, \tag{8}$$

$$\sin(\theta) = \frac{1}{2i}\left(z - \frac{1}{z}\right)\,, \tag{9}$$

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that turns them into rational functions of z on a complex circle; indeed, $d\theta = dz/(iz)$ and:

$$\int_0^{2\pi} F(\cos(\theta), \sin(\theta))\, d\theta = \oint_C \frac{f(z)}{iz}\, dz\,, \tag{10}$$

that can thus be integrated by the method of Residues. We will see in next lectures how other types of integrals can be similarly easily calculated thanks to residues.

A. Suggested readings

- "Residue Theory" from WikiBooks: `http://en.wikibooks.org/wiki/Complex_Analysis/Residue_Theory` (or `http://goo.gl/OMNH1w`).
- "*Sur la définition générale des fonctions analytiques, d'après Cauchy*", E. Goursat, Trans. Amer. Math. Soc. , **1**:14 (1900).

B. Exercises

1. Find the singularities and the corresponding residues of $\sin(2z)/z^6$, $\cos(z)/z^3$, $1/(1+z^2)$, $\cot(z)$ and $1/(1-e^z)$.
2. Calculate:

$$\oint_{|z|=1} \exp(1/z)\, dz\,, \qquad \oint_{|z-i\pi/2|=4.5} \frac{\exp(z)}{\cos(z)}\, dz\,, \qquad \oint_{|z-2-i|=5} \frac{z}{z^2-4z-5}\, dz\,, \tag{11a}$$

$$\oint_{|z-2i|=2} \frac{\sinh(z)}{2z-i}\, dz\,, \qquad \oint_{|z|=1/2} \frac{\cos(\pi z)}{z^5}\, dz\,, \qquad \oint_{|z|=1} \frac{30z^2-23z+5}{(2z-1)^2(3z-1)}\, dz\,. \tag{11b}$$

3. Calculate:

$$\int_0^{2\pi} \frac{1+\sin\theta}{3+\cos\theta}\, d\theta\,, \qquad \int_0^{\pi} \frac{d\theta}{k-\cos\theta}\,, \qquad \int_0^{2\pi} \frac{\cos\theta}{13-12\cos(2\theta)}\, d\theta\,. \tag{12}$$

C. Problems

1. Demonstrate the Residue Theorem.
2. What happens with the residue of a removable singularity?

D. "Café con Investigadores"

Dr. Elena del Valle, Don Carlos Sánchez Muñoz and Don Juan Pablo Restrepo Cuartas are three researchers at the Post-doctoral and Ph. D level in my group. In the framework of "Outreach Activity" that is part of EdV's funding by the European Union, they will present informally around a café (tea, orange juice and other popular drinks also available) their activity and what life looks like after the "master" for those wishing to pursue a scientific career. Elena has been awarded all the major research fellowship available in Europe: FPU (Spain), Newton fellowship (England), Humboldt fellowship (Germany), Marie Curie fellowship (Spain) and Rámon y Cajal fellowship (Spain). Carlos is starting a similar trajectory, presently at the FPI stage (Spanish funding) and Juan Pablo is undertaking Ph. D research as part of an international collaboration (EU funding). We will discuss the opportunities for young scientists in the drastic economic situation of today.

Drinks are offered, the event will be recorded.

Please register your attendance at `fabrice.laussy@gmail.com` (registration is mandatory) to arrange a suitable date and catering for everybody.