Mathematical Methods II
Handout 25. Fourier Series.

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For a function $U(t)$ defined on a finite compact support, say on $[-\pi,\pi]$, which has a derivative (the general condition is for piecewise continuous functions which have piecewise derivatives), one can define its Fourier Series $S(t)$ as:

$$S(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty}(a_j \cos jt + b_j \sin jt)\,, \tag{1}$$

where the coefficients a_j and b_j read:

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} U(t)\cos(jt)\,dt \quad \text{and} \quad b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} U(t)\sin(jt)\,dt\,. \tag{2}$$

This is a decomposition that describes periodic functions, as equivalently as considering $[-\pi,\pi]$ only, one can make copies of the functions over the entire line. The "complex Fourier Series" is defined as:

$$\tilde{S}(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}\,, \tag{3}$$

with, this time:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(t)e^{-int}\,dt\,, \tag{4}$$

for all $n \in \mathbf{N}$. To relate Eqs. (1) and (6) together, we expand the latter in trigonometric form through the Moivre formula which shows that:

$$c_0 = a_0/2\,, \quad a_n = c_n + c_{-n} \quad \text{and} \quad b_n = i(c_n - c_{-n})\,. \tag{5}$$

By rescaling $-\pi \le t \le \pi$ by L/π, we can stretch the function over the interval $[-L, L]$ for the variable $\tau = tL/\pi$, providing:

$$\tilde{S}(t) = \sum_{n=-\infty}^{\infty} c_n \exp\big(i\frac{\pi n t}{L}\big) \quad \text{with} \quad c_n = \frac{1}{2L}\int_{-L}^{L} U(\tau)e^{-i\dfrac{\pi n \tau}{L}}\,d\tau\,, \tag{6}$$

The terms $\omega_n = \frac{\pi n}{L}$ are called the "frequencies" and the set $\{\omega_n\}$ the "frequency spectrum". As $L \to \infty$, it is clear that the frequency spectrum forms a continuum, leading eventually to a continuous function also in the frequency space. This function is called the "Fourier transform" of the original function $\mathcal{F}[f(t)](\omega)$. One usually uses lighter notations than this. The way to compute the "coefficient" is now given in a beautiful symmetrical form of the transform itself:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt\,, \qquad f(t) = \int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega\,. \tag{7}$$

One speaks of the Fourier and the Inverse Fourier transforms. They are linked through the "symmetry" property: if $F(\omega) = \mathcal{F}(U(t))$, then $\mathcal{F}(F(t)) = U(-\omega)$.

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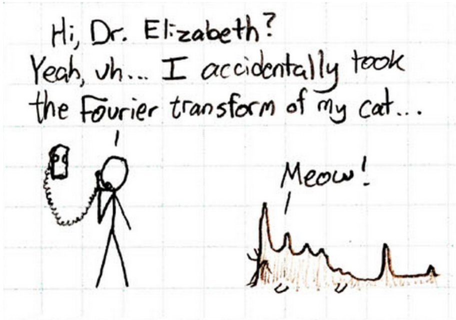

For instance, $\mathcal{F}[\exp(-|t|)](\omega) = \dfrac{1}{1+\omega^2}$. This is readily established by direct computation:

$$\mathcal{F}(\omega) = \int_{-\infty}^{\infty} e^{-|t|} e^{i\omega t}\, dt = \int_{-\infty}^{0} e^{(1+i\omega)t}\, dt + \int_{0}^{\infty} e^{(-1+i\omega)t}\, dt = \frac{1}{1+i\omega} e^{(1+i\omega)t}\Big|_{-\infty}^{0} = \frac{1}{1+\omega^2}\,. \tag{8}$$

Important properties of the Fourier transform that are useful for calculations are:

- Linearity: $\mathcal{F}(aU + bV) = a\mathcal{F}(U) + b\mathcal{F}(V)$.
- Scaling: $\mathcal{F}(U(at)) = \frac{1}{|a|}\mathcal{F}(\omega/a)$.
- Derivatives become products: $\mathcal{F}(U'(t)) = i\omega F(\omega)$.

Fourier transforms are useful in many applications. For intance, Schrödinger's equation $i\hbar\partial_t\psi(x,t) = -\frac{\hbar^2\nabla^2}{2m}\psi(x,t)$ is readily solved through a Fourier transform: $i\hbar\partial_t\psi(k,t) = \frac{(\hbar k)^2}{2m}\psi(k,t)$ and therefore $\psi(k,t) = \exp(-i\frac{\hbar k^2}{2m})\psi(k,0)$.

A. Suggested readings

- "Fourier Transform", E. W. Weisstein from MathWorld at `http://mathworld.wolfram.com/FourierTransform.html` or `http://goo.gl/Um8Lz`.
- `http://en.wikipedia.org/wiki/Fourier_transform`.
- `http://en.wikipedia.org/wiki/List_of_Fourier-related_transforms`.
- "The DFT "à Pied": Mastering The Fourier Transform in One Day.", Stephan Bernsee's Digital Signal Processing course at `http://www.dspdimension.com/admin/dft-a-pied/` or `http://goo.gl/uf6WJ`.

B. Exercises

1. Show that $U(t) = t/2$ for $-\pi \le t \le \pi$ extended periodically ($U(t+2\pi) = U(t)$) has the Fourier expansion $U(t) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}\sin(jt)$.
2. Compute the Fourier transform of the Heavyside $\Theta(t)$ function.
3. Compute the Fourier transforms of a constant, the Dirac δ function, the cos and sin functions, etc.
4. Compute the Fourier transform of a step function.

C. Problems

1. Compute the Fourier transforms of a Gaussian. Comment.
2. Show that $\frac{d^n F(\omega)}{d\omega^n} = \mathcal{F}((-it)^n U(t))$.
3. Moment theorem: Show that the moment $M_n = \int_{-\infty}^{\infty} t^n U(t)\, dt$ is given by $M_n = 2\pi F^{(n)}(0)/(-i)^n$.