# MÉTODOS MATEMÁTICOS II
## Lecture 8: Exponentials, logarithms, roots, trigonometric functions, hyperbolics and their inverses.

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Euler's formula is valid for complex arguments $e^{iz} = \cos z + i \sin z\,, \quad z \in \mathbf{C}$, as can be seen again by serie expansion (exp is holomorphic, hence analytic):

$$e^{iz} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \tag{1}$$

and defining the sine an cosine of the complex variable $z$ as the real and imaginary part of Eq. 1. This we can do since when $z$ is real, we recover the usual trigonometric functions.

From this follows $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$. We also define $\tan(z) = \sin(z)/\cos(z)$.

Most properties extend to the complex realm, such as $\cos^2 z + \sin^2 z = 1$, $\cos(-z) = \cos(z)$, $\sin(-z) = -\sin(z)$, $\tan(-z) = -\tan(z)$, etc. Others break down, e.g., $|\sin(z)|^2 = \sin^2 x + \sinh^2 y$ and $|\cos(z)|^2 = \cos^2 x + \cosh^2 y$. This shows that the complex sine and cosine are unbounded.

Hyperbolic functions are defined as:

$$\cosh(z) = \frac{e^z + e^{-z}}{2}\,, \qquad \sinh(z) = \frac{e^z - e^{-z}}{2} \tag{2}$$

with also $\tanh(z) = \sinh(z)/\cosh(z)$, etc. We now have $\cosh^2 z - \sinh^2 z = 1$. The cosh describes the catenary, that is, the curve that a hanging chain or cable assumes under its own weight when supported only at its ends.

The link between trigonometric and hyperbolic functions is through complex numbers:

$$\sin(iz) = i\sinh(z)\,, \quad \cos(iz) = \cosh(z)\,, \quad \tan(iz) = i\tanh(z)\,, \tag{3a}$$

$$\sinh(iz) = i\sin(z)\,, \quad \cosh(iz) = \cos(z)\,, \quad \tanh(iz) = i\tan(z)\,. \tag{3b}$$

Notations exist that are important to know, such as $\sec(z) = 1/\cos(z)$, $\csc(z) = 1/\sin(z)$, $\cot(z) = 1/\cot(z)$ for the secant, cosecant, cotangent, etc. They also exist for the hyperbolic case where they can become monstruous, e.g., $\mathrm{csch} = 1/\sinh$. More common is $\mathrm{sech} = 1/\cosh$.

More important than this terminology are inverse functions in the sense if $z = \sin(w)$, what is $w$ as a function of $z$? The answer is $\arcsin(z)$. Because trigonometric and hyperbolic functions are all periodic, they are many-to-one; hence their inverses are necessarily multivalued. The most important ones are (cf. Problem 5):

$$\arcsin(z) = -i\ln(iz + \sqrt{1-z^2})\,, \qquad \arccos(z) = -i\ln(z + i\sqrt{1-z^2})\,, \qquad \arctan(z) = -\frac{i}{2}\ln\left(\frac{1+iz}{1-iz}\right)\,. \tag{4}$$

The derivatives of all these functions should also be known:

$$(\sin(z))' = \cos(z)\,, \quad (\cos(z))' = \sin(z)\,, \quad (\tan(z))' = \sec(z)^2\,, \tag{5a}$$

$$(\cot(z))' = -(\csc(z))^2\,, \quad (\sec(z))' = \sec(z)\tan(z)\,, \tag{5b}$$

$$(\arcsin(z))' = 1/\sqrt{1-z^2}\,, \quad (\arccos(z))' = 1/\sqrt{1-z^2}\,, \quad (\arctan(z))' = 1/(1+z^2)\,. \tag{5c}$$

### A. Suggested readings

- Online encyclopedias, such as `http://en.wikipedia.org/wiki/Inverse_hyperbolic_function` or `http://mathworld.wolfram.com/InverseHyperbolicFunctions.html` and references therein.
- `http://laussy.org/wiki/MMII`

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**PROBLEMS TO RETURN (by "February 18" the latest)**

Each problem will be marked on a scale of 2.5 points on a total of 10. You can do as many as you want but it is recommended to chose four and focus on them, as this is enough to get the maximum mark. The note will be given by the four highest notes if you study more than that.

## I. PROBLEM 1

Find the complex numbers whose additive inverses and multiplicative inverses are the same.

*First you have to figure out what an "additive inverse" and a "multiplicative inverse" is, if you don't know it. This is part of a problem, to understand what is asked. Sometimes this is the most difficult part, in particular as science is redundant with vague or even contradictory definitions and concepts.*

## II. PROBLEM 2

What is the area of the Mandelbrot set in the region $\mathrm{Re}(z) \geq 0$.

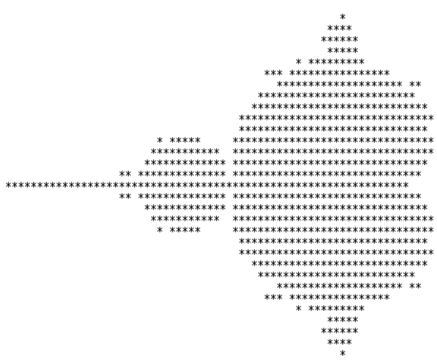

FIG. 1: *The Mandelbrot set as first pictured, in 1978 with the tools available then, by the Mathematician Robert Brooks.*

## III. PROBLEM 3

Prove that an harmonic function that depends only on the distance $r$ from a given point is of the form $a \log r + b$ where $a$ and $b$ are constants.

## IV. PROBLEM 4

Study $\lim_{n\to\infty} n\left(\frac{1+i}{2}\right)^n$ and $\lim_{n\to\infty} ni^n$.

## V. PROBLEM 5

Prove the identities (4) [recto]. Obtain the corresponding expressions for arcsinh, arccosh and arctanh.

## VI. PROBLEM 6

Prove the identities (5) [recto].

## VII. PROBLEM 7

Find the images of the shaded areas under the transformations $z \to z^2$ and $z \to 1/z$.

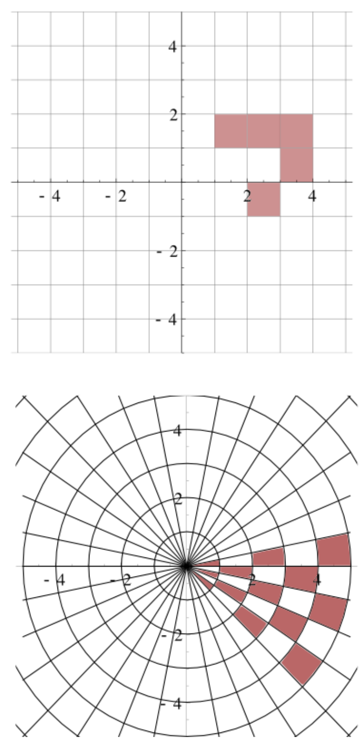


FIG. 2: Some patterns in the complex plane.

## VIII. PROBLEM 8

The dynamics of a damped harmonic oscillator reads:

$$m\ddot{x} + \gamma\dot{x} + kx + F = 0 \tag{6}$$

with $m$ the mass, $\gamma$ the decay and $k$ the spring constant some constants $\in \mathbf{R}$, $F$ a driving force and $x$ the oscillator's displacement.

This equation can be solved exactly for any driving force using the solutions $z(t)$ to the unforced equation. Show this and study the solutions.