MÉTODOS MATEMÉTICOS II
Lecture 11: Integrals in the complex plane.

Fabrice P. Laussy[1]

[1]*Departamento de Física Teórica de la Materia Condensada, Universidad Autónoma de Madrid*[*]

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The Riemann integral is the limit of Riemann-sums:

$$\sum_{k=1}^{N} f(x_k)(x_k - x_{k-1}) \tag{1}$$

as $N \to \infty$ and $x_k - x_{k-1} \to 0$. This approximates the area below the curve $f(x)$. This is used to define complex integrals (that is, integration of complex functions of the complex variable), taking the limit of $\sum_{k=1}^{N} f(z_k)(z_k - z_{k-1})$, but now with $z_k \in \mathbf{C}$. We find, for instance, $\int_0^i z dz = \lim \frac{i}{N}\Delta z + \frac{2i}{N}\Delta z + \cdots + \frac{Ni}{N}\Delta z = \lim_{N\to\infty} \sum_{k=1}^{N} \frac{ik}{N}\frac{i}{N} = -1/2$ with Δz one of the little path of integration as we go from 0 to i along the y axis.

Formal results of integration still hold, in the form:

$$\int z^n \, dz = \frac{z^{n+1}}{n+1}\,, \quad \int \frac{dz}{z^n} = \begin{cases} -\dfrac{1}{(n-1)z^{n-1}} & \text{if } n > 1\,, \\ \log(z) & \text{if } n = 1\,. \end{cases} \tag{2}$$

In particular, $\int z dz = z^2/2$ and we recover the above results. Also properties like $\int f + g = \int f + \int g$, etc., still hold, again, being properties (like here "linearity") of the operator $\int$ itself rather than of the real plane. One such transposition from the real to the complex plaine is not straightforward, namely, $\int_a^b f = \int_a^c f + \int_c^b f$, is not completely obvious at first sight, with a, b and $c \in \mathbf{C}$, for instance if these points do not fall on the same line with c in between. This is because, like for the case of derivability, the notion of "path" in complex calculus is an important one given the new degrees of freedom of the variables (from a line to a plane), and in general we need to specify which path is taken, although we will see that for holomorphic function, this is not always necessary, which allows to write identities such as Eq. (2). The notation that specifies the path of integration is $\int_{\mathcal{C}} f(z)dz$ where $\mathcal{C}$ is the said path, that can be anything (a line, a curve, continuous or not, crossing itself, etc.)

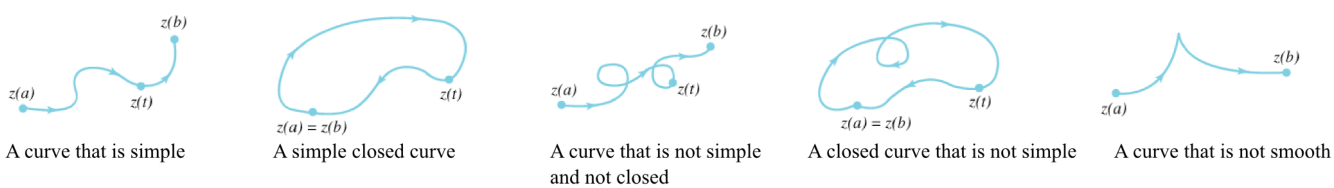

FIG. 1: Terminology of the various types of paths one encounters in complex integration.

For the case of a closed path, the notation $\oint$ is used. Paths that are not smooth can be broken into various paths on which the integration is computed piecewise, and the final result obtained by summing them up. When such a path is specified, we speak of a *definite integral*. Otherwise the integral is "indefinite".

Note that we could have computed $\int_0^i z dz$ as $\frac{1}{i}\int_0^i (iy)dy = \frac{1}{i^2}\int_0^1 (iy)d(iy) = -1/2$, the integral as function of iy being formally that already calculated in Eq. (2). Here we have simply changed the variable, linking the integrals through the two paths as:

$$\int_{\mathcal{C}} f(z)\, dz = \int_{\mathcal{D}} f(g(\xi))g'(\xi)\, d\xi \tag{3}$$

with $\mathcal{D}$ the path followed by ξ. This is true provided g is continuous and its derivative g' is continuous on $\mathcal{D}$.

[*]Electronic address: fabrice.laussy@gmail.com

If the new path is a parametric function of a single variable t, we can restore the lower/upper bounds notations. For example, $I = \int_{\mathcal{C}} \mathrm{Re}(z)\, dz$ where $\mathcal{C}$ is the straight line from 0 to $1+2i$ can be computed through $\mathcal{D} = \{z\,,\ z(t) = t(1+2i)\,,\ 0 \le t \le 1\}$ and $I = \int_0^1 \mathrm{Re}(z(t))z'(t)dt = \int_0^1 t(1+2i)dt = i + 1/2$.

The complex integral is complex in general. A measure of its magnitude is given by its modulus, that can be easily bounded:

$$\left|\int_{\mathcal{C}} f(z)\, dz\right| \le ML \tag{4}$$

with M such that $|f(z)| \le M$ for all $z \in \mathcal{C}$ and L the length of $\mathcal{C}$. This can be used to quickly check a result.

Complex calculus allows, here too, to obtain otherwise complicated results easily. For instance, $\int_0^{\pi/2} e^{ax}\cos x\, dx = \frac{e^{a\pi/2}-a}{a^2+1}$. It is also important for the integrals of rational functions, $P(x)/Q(x)$, which can always be obtained in closed form. If $\deg(P) < \deg(Q)$, there exists $c_{ij} \in \mathbf{C}$ such that:

$$\frac{P}{Q} = \sum_{i=1}^{k}\sum_{j=1}^{m_j} \frac{c_{ij}}{(x-\alpha_i)^j} \tag{5}$$

that can easily be computed through Eq. (2). For instance, by the property just announced, there exists A, B, C and D such that $\frac{1}{1+x^4} = \frac{A}{x-(1+i)/\sqrt{2}} + \frac{B}{x-(1-i)/\sqrt{2}} + \frac{C}{x+(1-i)/\sqrt{2}} + \frac{D}{x+(1+i)/\sqrt{2}}$, since the coefficients in the denominators are the roots of $1 + x^4 = 0$ (no multiplicity). By multiplying both sides by the denominator of each expression of the rhs in turn and evaluating the left side (that cancels the pole), we find $A = \sqrt{2}(-1-i)/8$, $B = \sqrt{2}(-1+i)/8$, $C = \sqrt{2}(1-i)/8$ and $D = \sqrt{2}(1+i)/8$. This can be further written as $\log(a+ib) = \frac{1}{2}\log(a^2+b^2) + i\arctan(b/a)$; beside, one can use $\arctan x + \arctan 1/x = \pm\pi/2$ depending on the sign of x, to have the variable on the denominator; in this way, we arrive at:

$$\int \frac{dx}{x^4+1} = \frac{\sqrt{2}}{8}\log\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + \frac{\sqrt{2}}{4}\Big(\arctan(x\sqrt{2}+1) + \arctan(x\sqrt{2}-1)\Big)\,. \tag{6}$$

Whenever dealing with complex function, it is tempting to link them with the underlying real functions of real variables $f(z) = u(x,y) + iv(x,y)$. This gives, for the general case:

$$\int_{\mathcal{C}} f(z)\, dz = \int_{\mathcal{C}} (u+iv)(dx+idy) = \int_{\mathcal{C}} u\, dx - v\, dy + i\int_{\mathcal{C}} v\, dx + u\, dy\,. \tag{7}$$

Green's theorem provides such a link between the contour integral (on a line that closes on itself) and the surface integral through the surface $\mathcal{R}$ enclosed by $\mathcal{C}$:

$$\oint_{\mathcal{C}} P\, dx + Q\, dy = \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy\,. \tag{8}$$

The functions $P(x,y)$ and $Q(x,y)$ must be continuous with partial derivatives also continuous on $\mathcal{R}\cup\mathcal{C}$. We will see next lecture the properties of complex integrals for holomorphic functions.

A. Exercises

1. Compute $\int_0^1 (x-i)^3\, dx$ (x real) and $\int_0^{\pi/3} e^{i\theta}\, d\theta$ (θ real).
2. Compute $\int_{\mathcal{C}} z\, dz$ and $\int_{\mathcal{C}} z^*\, dz$ for $\mathcal{C}$ the path from 0 to $1+i$ that is either straight between these two points, or through an horizontal and vertical segment.
3. Same as 2. for $\mathcal{C}$ specified by $z(t) = t^2(1+i)$ and $z(t) = t(t+i)$.
4. Same as 2. but integrating z^2 and e^z vs. z^{2*} and e^{z^*}.
5. Compute $\int_{\mathcal{C}} z^2\, dz$ on the circle $|z| = 4$.

B. Problems

1. Fill in the necessary steps to establish Eq. (6).
2. Represent the function $1/(1+x^6)$ between -2 and 2 and compute the area below this curve.
3. Study the continuous deformation of a path, say, $z(t) = t + ia\sin(t)$ for $0 \le t \le \pi$ for the parameter $a \in \mathbf{R}$ and study the integrals of, e.g., $\mathrm{Re}(z^2)$ and $|z|^2$ as function of a and of z and z^2. Compare and discuss.