Mathematical Methods II
Handout 11: Conformal Mappings.

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A *Conformal mapping* is a transformation that preserves the angles. In 2D
For the function $f(z) = u(x, y) + iv(x, y)$, the Jacobian $J_f(x, y)$ reads:

$$J_f(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}\,, \tag{1}$$

The Jacobian is the matrix of transformation from a set of coordinates (those derivating) to a new one (those being derivated). Its determinant represents the ratio of volumes in the old and the new coordinates. It is also commonly written as $\frac{\partial(u,v)}{\partial(x,y)}$. If the Jacobian (determinant) is not zero, the transformation is bijective. In the case of an holomorphic function f, from the Cauchy–Riemann equations, we get straightforwardly:

$$\frac{\partial(u,v)}{\partial(x,y)} = |f'(z)|^2\,, \tag{2}$$

Points where $(\partial_x u)^2 + (\partial_x v)^2 = 0$ are called *critical points.*

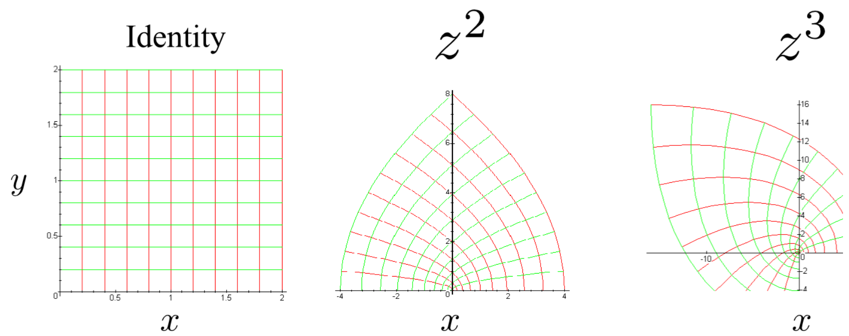

FIG. 1: Conformal mapping through $z \to z^2$ and z^3: the transformation is angle preserving.

The stretching in area of the square is roughly $|f'|^2$.
The *Möbius transform* is given by:

$$w = \frac{az+b}{cz+d} \tag{3}$$

with a, b, c and $d \in \mathbf{C}$ such that $ad \neq bc$. It maps lines and circles into other lines and circles, since:

$$w = \frac{az+b}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c}\frac{1}{cz+d} \tag{4}$$

is a combination of linear transform $z_1 = cz + d$ followed by an inverse $z_2 = 1/z_1$ then a scaling by $(bc - ad)/c$ and finally transation by a/c, each of which in isolation preserves circles.

There exists a unique Möbius transformation that maps three distinct points z_i, $i \in \{1, 2, 3\}$, onto three distinct points w_i, respectively. An implicit formula for the mapping is given by the equation:

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}\,. \tag{5}$$

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For example, the Möbius transform that maps $\{-i, 1, i\}$ into $\{-1, 0, 1\}$ leads to $(z+i)(1-i)(z-i)(1+i)/(w+1-w+1) = i(1-z)/(1+z)$ It is plotted on Fig. 2.

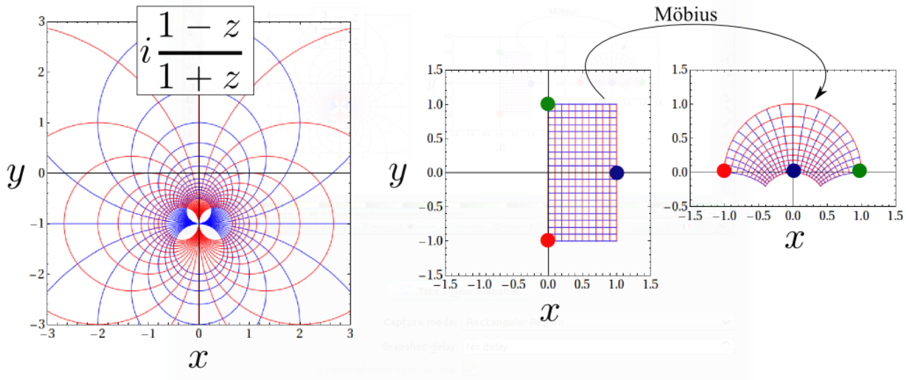

FIG. 2: The Möbius transform that maps $\{-i, 1, i\}$ to $\{-1, 0, 1\}$; the full transform on the left and the transform of the rectangular patch on the right.

The *Schwarz-Christoffel mapping* is a more general transform, that defines a conformal mapping of the upper half-plane onto the interior of a simple polygon.

The *Riemann mapping theorem* states that if U is a non-empty simply connected open subset of $\mathbf{C}$ (not $\mathbf{C}$ itself), then there exists a bijective and holomorphic mapping f from U onto the open unit disk $\{z \in \mathbf{C}, |z| < 1\}$.

The interest of such transforms is that they transport the harmonic properties. Illustratively, if f is a bijective conformal transform between two complex domains, and ϕ an analytic function of the variables x, y, we have $\nabla^2_{x,y}\phi = |f'(z)|^2 \nabla^2_{u,v}[\phi(f^{-1}(u,v))]$, where the rhs involves the image space, in the new coordinate u, v. If we can solve a problem in one geometry, we have solved it in all geometries to which we can map conformally.

A. Suggested readings

- "Möbius Transformations Revealed", user jonathanrogness on YouTube at `http://www.youtube.com/watch?v=JX3VmDgiFnY`.
- "Möbius Transformations Revealed", D. N. Arnold and J. Rogness, Notices of the AMS, 55 1226 (2008), online at `http://www.ima.umn.edu/~arnold/papers/moebius.pdf`.
- "Visual complex analysis", T. Needham, Oxford University Press (1998) and at `http://usf.usfca.edu/vca`.
- "Indra's Pearls", D. Mumford, C. Series and D. Wright, Cambridge University Press (2002).
- "Breakthrough in Conformal Mapping", J. Case, SIAM News, 41(1) (2008) online at at `http://siam.org/pdf/news/1297.pdf`.
- The Wolfram Demonstrations Project has various applets to experience conformal mapping, e.g., M. Trott shows a tunable conformal mapping $z \to (1-a)(\alpha_0 + \alpha_1 z + \alpha_2 z^2) + a\cos(\beta_0 + \beta_1 z + \beta_2 z^2)$ at `http://demonstrations.wolfram.com/ConformalMaps`

B. Exercises

1. Study the mapping of $z \to z^n$ (n integer) and $e^{\pi z/a}$.
2. Study the mapping of the triangle delimited by $x = 1$, $y = 1$ and $x + y = 1$ by the transforms $z \to z^2$ and $1/z$.
3. Find the Möbius transform that maps the points $\{-1-i, 0, i\}$ onto the points $\{2i, 0, -1+i\}$, respectively.
4. Show that the unit circle in Fig. 2 maps to half the complex plane. Which one?

C. Problems

1. Use the Möbius transform to find a bijective conformal mapping between any two subsets of the complex planes when the subsets are disks or half planes delineated by a line.
2. Show that the Jacobian matrix of the transformation is everywhere a scalar times a rotation matrix.