Mathematical Methods II
Handout 24. Applications of Residue Theory.

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We have already seen how complex integration allows for the evaluation of rational functions in the polar form. A similar technique can be computed for some integrals of the type $\int_{-\infty}^{+\infty} f(x)\,dx$. Such integrals may not exist, for instance, $\int_{-\infty}^{\infty} x\,dx = \frac{1}{2}x^2\Big|_{-\infty}^{+\infty}$ is undefined. Taking the limit of always well-defined integrals, we introduce the "Cauchy Principal Value":

$$\mathrm{PV}\int_{-R}^{R} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx\,, \tag{1}$$

which in the case above gives a well-defined and finite result: PV $\int_{-\infty}^{\infty} x\,dx = 0$. These integrals turn out to be useful in physics whereas the undefined version is merely a frustration.

A first nontrivial application of Residues is to compute integrals of rational functions P/Q with $\deg(Q) \ge \deg(P)+2$ and $Q(x) \neq 0$ for all $x \in \mathbf{R}$:

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2i\pi \sum_{j=1}^{k} \mathrm{Res}\left[\frac{P}{Q}, z_j\right] \tag{2}$$

where z_j for $1 \le j \le k$ are the poles in the upper complex plane $\mathrm{Im}(z) \ge 0$.

For instance, we can readily compute in this way:

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)} = \pi/6\,. \tag{3}$$

To find this, we introduce:

$$f(z) = \frac{1}{(z+i)(z-i)(z+2i)(z-2i)}\,,$$

which has residues in the upper plane $\mathrm{Res}_{z=i}(f) = -i/6$ and $\mathrm{Res}_{z=2i}(f) = i/12$, so that, the integral is $2i\pi(-i/6 + i/12)$.

If $Q(x)$ has poles on the real axis, the theorem reads, in the same conditions as above:

$$\mathrm{PV}\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2i\pi \sum_{j=1}^{k} \mathrm{Res}\left[\frac{P}{Q}, z_j\right] + i\pi \sum_{j=1}^{l} \mathrm{Res}\left[\frac{P}{Q}, x_j\right] \tag{4}$$

with $x_1, \ldots, x_l$ simple poles on the real axis.

The argument principle: For a meromorphic function f in a simply connected domain $\mathcal{D}$ enclosed by $\mathcal{C}$ such that f has no zero or pole on $\mathcal{C}$, then:

$$\frac{1}{2i\pi}\oint_{\mathcal{C}} \frac{f'(z)}{f(z)}\,dz = Z_f - P_f\,, \tag{5}$$

with Z_f the number of zeros of f in $\mathcal{D}$ and P_f the number of poles.

Rouché's theorem: For f and g two meromorphic functions in a simply connected domain $\mathcal{D}$ enclosed by $\mathcal{C}$, with no zeros or poles on $\mathcal{C}$, if $|f(z)+g(z)| < |f(z)| + |g(z)|$ is true for all $z \in \mathcal{C}$, then $Z_f - P_f = Z_g - P_g$. (This theorem will be demonstrated in the "practicas").

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A. Suggested readings

- "Some Applications of the Residue Theorem", Pawel Hitczenko, `http://people.math.gatech.edu/~cain/winter99/supplement.pd` or `http://goo.gl/LNO9m9`.
- "More Applications of Residues", Bernd Schröder, 347 slides at `http://www2.latech.edu/~schroder/slides/comp_var/21_more_on_residues.pdf` or `http://goo.gl/ZpHOK4`.

B. Exercises

1. Evaluate

$$\int_0^\infty \frac{1}{1+x^4}\,dx, \qquad \int_{-\infty}^\infty \frac{dx}{x^2+16} \quad \text{and} \quad \int_{-\infty}^\infty \frac{dx}{(x^2+1)^2(x^2+4)}\,.$$

2. Evaluate

$$\int_{-\infty}^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

where $a > 0$ and $b > 0$.

3. Evaluate

$$\int_{-\infty}^\infty \frac{dx}{x^3+x}\,, \quad \int_{-\infty}^\infty \frac{xdx}{x^3+1} \quad \text{and} \quad \int_{-\infty}^\infty \frac{dx}{x(x-1)(x-2)}\,.$$

C. Problems

1. Compute PV $\int_{-\infty}^\infty \frac{\sin x}{x}\,dx\,.$
2. Prove Rouché's theorem (hint: introduce $w(z) = \ln(f(z)/g(z))$ and its derivative in the argument principle).

D. Continuous Examination

To return by 22 of April 2014.
Calculate (two points each):

$$\sum_{n=1}^\infty \frac{1}{n^4}\,, \tag{6}$$

$$\int_{-\infty}^\infty \frac{x^2}{x^4-1}\,dx\,, \tag{7}$$

$$\int_0^{2\pi} \frac{d\theta}{(1+3\cos^2\theta)^2}\,. \tag{8}$$

Provide the counterpart (statement and proof) of Eq. (2) for integrals of the type (four points):

$$\int_{-\infty}^\infty \frac{P(x)}{Q(x)}\cos(\alpha x)\,dx \qquad \text{and} \qquad \int_{-\infty}^\infty \frac{P(x)}{Q(x)}\sin(\alpha x)\,dx\,. \tag{9}$$