# Mathematical Methods II
# Handout 6: Continuity (for the mathematician).

Fabrice P. Laussy
*Departamento de Física Teórica de la Materia Condensada, Universidad Autónoma de Madrid**
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We will take the advantage of the question of continuity of complex functions to approach the subject from a more Mathematical viewpoint than is usual for a physicist, with an inclination to rigorous proofs such as those that a Mathematician would desire.

**Definition:** A topology on a set $X$ is a pair $(X, \mathcal{J})$ with $\mathcal{J}$ a subset of $\mathcal{P}(X)$ containing at least $\emptyset$ and $X$ and which is closed under the formation of arbitrary unions and finite intersections.

The members of $\mathcal{J}$ are called open sets.

**Definition:** A function $f$ is continuous if the inverse image of every open set is open.

This is the exact Mathematical definition. We will focus on the particular case of metric spaces. Here it is enough to deal with "open balls":

**Definition:** An "open ball" $B(a, r)$ of radius $r$ centered on $a$ is the set of points $B(a, r) = \{z \in \mathbf{C} \mid |z, a| < r\}$.

An open set is any union of open balls.

**Definition:** A neighbourhood $\mathcal{V}$ of a point $z$ is a set such that there exists an open set $O$ that contains $z$ and is included in $\mathcal{V}$.

**Proposition:** A function $f$ is continuous at the point $z$ iff for any neighborhood $\mathcal{V}$ of $f(z)$, there is a neighborhood $\mathcal{U}$ of $z$ such that $f(\mathcal{U}) \subset \mathcal{V}$.

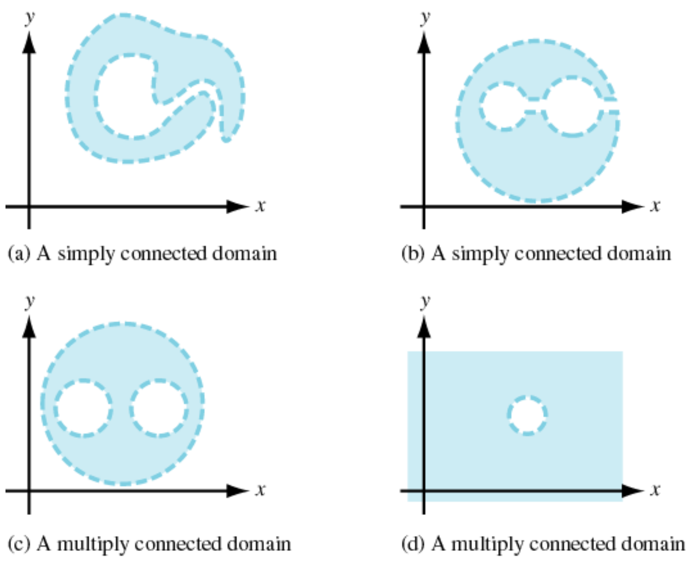


FIG. 1: A domain $D$ is an open set such that any two points can be connected by a broken line segment in $D$. A domain $C$ is convex if any two points can be joined by a straight line in $C$. A domain is path-connected if a path can be drawn between any two points in the space. It is furthermore simply connected if every path between two points can be continuously transformed, staying within the space, into any other path while preserving the two endpoints in question.

*Electronic address: fabrice.laussy@gmail.com


### A. Suggested readings

- Browse `http://en.wikipedia.org/wiki/List_of_general_topology_topics` (http://goo.gl/BHv3n) and see how many Mathematical concepts of topology you are now familiar with; explore the others.
- "Principles of Mathematical Analysis", 3rd Edition, Rudin, McGraw-Hill (1976).

### B. Exercises

1. Show that the set where the opens are the empty set and the entire space is a topology (it is called the "trivial topology").
2. Show that $\mathcal{P}(X)$ is a topology.
3. Show that if $X = \{1, 2, 3, 4\}$, then $\mathcal{T} = \{\{\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$ is a topology.
4. Show that if $X = \mathbf{Z}$, the set of integers, and $\mathcal{N}$ is the set of all finite subsets of the integers plus $\mathbf{Z}$ itself, then $\mathcal{N}$ is not a topology.

### C. Problems

1. Prove that the topological and Weierstrass definition of continuity are equivalent.
2. Discuss the difference between these two definitions of continuity:

$$(\forall z \in \mathbf{C})(\forall \varepsilon > 0)(\exists \eta > 0)(\forall w \in \mathbf{C})(|z - w| < \eta \;\Rightarrow\; |f(z) - f(w)| < \varepsilon)\,,$$

   and:

$$(\forall \varepsilon > 0)(\exists \eta > 0)(\forall z \in \mathbf{C})(\forall w \in \mathbf{C})(|z - w| < \eta \;\Rightarrow\; |f(z) - f(w)| < \varepsilon)\,.$$

3. Can you think of a function that is nowhere continuous?