# MÉTODOS MATEMÁTICOS II
## Lecture 9: Complex Potentials.

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Potential theory studies functions $\Phi$ that satisfy Laplace's equation $\nabla^2\Phi = 0$. In two dimensions, there is a tight link with complex analysis: for such a $\Phi$, there exists $F$ holomorphic such that $F(z) = \Phi(x, y) + i\Psi(x, y)$ with $\Psi$ also harmonic. The function $F$ is called the "*complex potential*" of $\Phi$. $\Psi$ has the physical meaning of "lines of forces", since the curves $\Psi(x, y) = \mathrm{cst}$ are orthogonal to the equi-potential lines $\Phi(x, y) = \mathrm{cst}$.

For example, the potential between two parallel plates situated on the $x$ axis at $\alpha$ and $\beta$ and with an applied potential $\Phi_\alpha$ and $\Phi_\beta$ gives rise to the real potential:

$$F(z) = \left(\frac{\Phi_\alpha - \Phi_\beta}{\alpha - \beta} + \frac{\Phi_\alpha - \Phi_\beta}{\alpha - \beta}\right) z + \frac{1}{2}\left((\Phi_\alpha + \Phi_\beta) - (\Phi_\alpha - \Phi_\beta)\frac{\alpha + \beta}{\alpha - \beta}\right) . \tag{1}$$

The equipotentials are lines horizontal to the plates, $x = \mathrm{cst}$, while lines of forces are those perpendicular, joining them, $y = \mathrm{cst}$.

Another example in polar representation. The Laplacian in such coordinates reads:

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} . \tag{2}$$

giving rise to the complex potential:

$$F(z) = \frac{\Phi_\alpha - \Phi_\beta}{\ln(r_\alpha - r_\beta)}\ln(z) + \frac{1}{2}\frac{\Phi_\alpha + \Phi_\beta}{\Phi_\alpha - \Phi_\beta}\frac{\ln(r_\alpha r_\beta)}{\ln(r_\alpha/r_\beta)} \tag{3}$$

with equipotentials $r = \mathrm{cst}$ circles between the cylinders and lines of forces $\theta = \mathrm{cst}$ the perpendiculars joining them.

A strength of potential theory is that any linear superposition of solutions is also a solution. The complex potential of one line charge at point $z_0$ being $F_1(z) = \alpha\ln(z - z_0)$, that for two such lines, say of opposite charges $\alpha$ and at $z_0$ and $-z_0$ reads:

$$F(z) = \alpha[\ln(z - z_0) - \ln(z + z_0)] \tag{4}$$

The equipotentials are then $\mathrm{Re}(F(z)) = \mathrm{cst}$, i.e.,

$$\left|\frac{z - z_0}{z + z_0}\right| = \mathrm{cst} \tag{5}$$

These are circles. The lines of force are given by $\mathrm{Im}(F) = \arg\frac{z-z_0}{z+z_0} = \mathrm{cst}$, which reduces to $\theta_1 - \theta_2 = \mathrm{cst}$ which are also circles (cf. exercise 2).

We have considered the electric field but the notion of a complex potential applies to any 2D potential, such as for the heat equation. We will consider further applications in fluid mechanics, namely, for a 2D steady, non-viscous and incompressible fluid, which allows the fluid velocity to derive from a potential:

$$V_x = \frac{\partial\Phi}{\partial x} , \quad V_y = \frac{\partial\Phi}{\partial y} . \tag{6}$$

The holomorphic $F(z) = \Phi + i\Psi$ provides (from $F'(z) = \partial_x\Phi - i\partial_y\Psi$) the complex velocity as:

$$V_x + iV_y = F'(z)^* . \tag{7}$$

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The speed at a given point is $|F'(z)|$. Points where $F'(z) = 0$ are "stationary points", where the fluid doesn't move. The complex potential $z^2$ describes flow past a corner. The potential

$$F(z) = z + \frac{1}{z}\,. \tag{8}$$

yields, passing by the polar form, $F = (r + \frac{1}{r})\cos\theta + i(r - \frac{1}{r})\sin\theta$, streamlines as:

$$\left(r - \frac{1}{r}\right)\sin\theta = \mathrm{cst} \tag{9}$$

which are shown in Fig. 1. This describes the important problem of a flow past a cylinder.

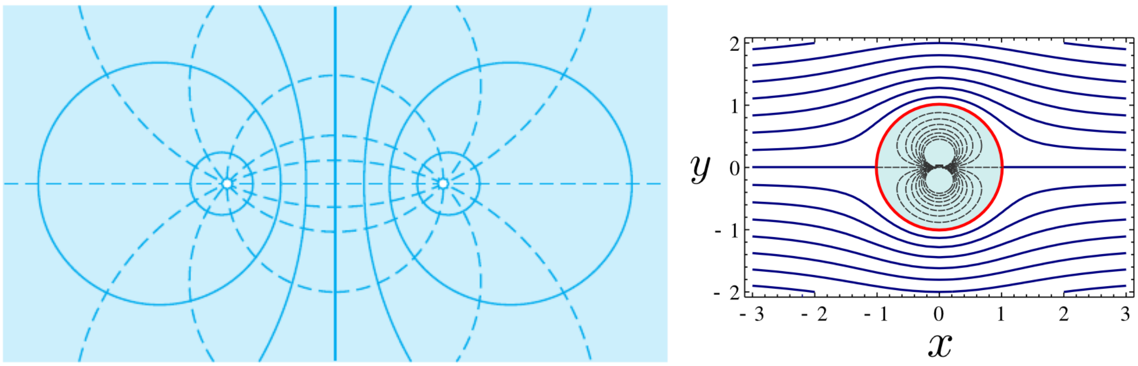


FIG. 1: Left: Complex potential from two lines perpendicular to the plane, giving rise to equipotentials (solid) and lines of force (dashed); Right: Potential flow from the complex potential $z + 1/z$. The solution $r^2 = 1$ is used as the obstacle, the outer lines represent the flow of an ideal fluid past it, the inner (dashed) lines are solutions not used here.

### A. Suggested readings

- "2D potential flow" Chap. 6 of "Fluid Mechanics", R. Fitzpatrick, p. 101 at `http://goo.gl/zy8aj`.
- "The Laplacian in polar coordinates", Zhi Lin, at `http://goo.gl/TvNto`.
- "D'Alembert's paradox", for example on wikipedia, at `http://goo.gl/en0FM`.
- `http://laussy.org/wiki/MMII`

### B. Exercises

1. Use the result of the Problems below to check that the real and imaginary parts of $1/z$ are harmonic.
2. Check that the potential of Eq. (4) gives rise to that show in Fig. 1.
3. Study the complex potential $F(z) = iz^3$.

### C. Problems

1. Show that the Polar representation of the Cauchy-Riemann equations is $(\partial f/\partial r) = (1/ir)(\partial f/\partial\theta)$ [cf. handout 5].
2. Find the temperature field around a long thin wire of radius $r_1 = 1$mm that is electrically heated to $T_1 = 100°$ and is surrounded by a circular cylinder of radius $r_2 = 100$mm, kept at temperature $T_2 = 20°$.
3. Show that $F(z) = \arccos z$ defines the potential of a slit. How about arccosh?