# Mathematical Methods II
# Handout 12: Conformal Mappings 2.

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(Dated: February 18, 2014)

Each holomorphic function, and all their possible combinations, give rise to conformal mapping. One can go through a safari of the various stretching of the complex plane. Such as:

*a.* $\exp(z)$ — The function $\exp(z)$ is a one-to-one mapping between the horizontal strip $-\pi < y \le \pi$ onto the complex plane $\mathbf{C} - \{0\}$. The patch of the grid $\{z = x + iy : a < x < b \text{ and } c < y < d\}$ transforms into $\{\rho e^{i\theta} : e^a < \rho < e^b \text{ and } c < \theta < d\}$.

*b.* $(e^z - i)/(e^z + i)$ — This transforms the horizontal strip $0 < y < \pi$ onto the disk $|z| < 1$. The $x$-axis is mapped onto the lower semicircle bounding the disk and the $y = \pi$ axis is mapped onto the upper semicircle. This follows from the decomposition $z \to \exp(z) \to (z - i)/(z + i)$ with the latter Möbius transform mapping the upper half-plane $\Im(z) \ge 0$ onto the disk $|z| < 1$.

*c.* $z^n$ — This maps the infinite sector of angle $\pi/m$ to the complex plane.

*d.* $\sin(z)$ — This maps the vertical strip $-\pi/2 < x \le \pi/2$ into the complex plane. Since $\sin(z) = \sin x \cosh y + i \cos x \sinh y$, we find:

$$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1\,, \qquad \frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1\,, \tag{1}$$

showing that the Cartesian grid transforms into a net of hyperbolas and ellipses.

*e.* $z + 1/z$ From the polar coordinates: $w = z + 1/z = (r + 1/r)\cos\theta + (r - 1/r)\sin\theta)$, this maps centered circles $r = \text{cst}$ into ellipses.

Etc.

*f. Schwarz–Christoffel mapping* — The function

$$f(z) = c\int \frac{dz}{(z - a_1)^{k_1}(z - a_2)^{k_2}\dots(z - a_n)^{k_n}} + C,$$

with $a_1 < a_2 < \cdots < a_n$ and $k_j$ real numbers such that $|k_j| \le 1$, maps the real axis and the upper half-plane conformally onto the closed area bounded by a broken line. When $z$ moves on the real axis, $f(z)$ moves along the broken line so that the direction turns the amount $k_j\pi$ anticlockwise each time time $z$ passes a point $a_j$.

*A colonel from a top secret military research institution comes to a math department, and asks to find a conformal map from an equilateral triangle onto the upper half-plane. They tell him. A week later he comes again and asks about a conformal map of a square onto the upper half-plane. They tell him. Next time he comes and asks about regular pentagon and hexagon (which is much harder). The mathematicians start to suspect something… They ask him: "What is your ultimate goal?" He replies: "Well, I think I can tell you, though this is a secret research. We are trying to find a conformal map of a disc onto the upper half-plane, by approximating the disc by regular polygons with many sides!"*

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Solution to the class blitz problem: *If $f$ is continuous at $z_0$, then there exists a neighborhood of $z_0$ where it is bounded.* Proof: $f$ being continuous at $z_0$ means that there exists $\delta > 0$ such that for all $z$ for which $|z - z_0| < \delta$, $|f(z) - f(z_0)| \le \epsilon$ for an $\epsilon > 0$. In this neighborhood, $|f(z)| = |f(z) - f(z_0) + f(z_0)| \le |f(z) - f(z_0)| + |f(z_0)| \le \epsilon + |f(z_0)|$, i.e., it is bounded).

## A. Exercises

Represent the mapping of a patch of the plane through the transformations $(e^z - i)/(e^z + i)$, $\sqrt{z}$ and $\cos(z)$.

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### B. Suggested readings

- "Dictionary of Conformal Mapping" at `http://goo.gl/12wN7D`.
- "The Electric Field in Various Circumstances", R. P. Feynman, the Feynman Lectures on Physics, Volume II (`http://www.feynmanlectures.caltech.edu/II_toc.html`), Chapters 6 and 7.
- "Numerical conformal mapping software: zipper", Donald E. Marshall at `http://www.math.washington.edu/~marshall/zipper.html` (or `http://goo.gl/pJRjup`).
- Conformal mapping of photos and the "Little Planet" effect, at, e.g., `http://goo.gl/iwnKvw`.

### C. Project

Study the Joukowsky transform $z \to z + 1/z$ for off-centered circles (concretely consider the images of circles centered at $ia$ with radius $\sqrt{1+a^2}$ for $0 < a < 1$ and see how the shape this takes evokes possible applications with Laplace's equation).

### D. Continuous Examination

*(To return by 3rd of March)*

#### I. Homotopic Möbius (5 pts)

Represent on paper the transformation of the grid $-3 < x \leq 3$ and $-\pi < y \leq \pi$ by the complex function $f_t(z) = (1-t)\frac{1}{z} + t\exp(z)$ for

$$t = \frac{}{10}\,.$$

#### II. General Möbius (5 pts)

Show that any Möbius transform can be decomposed as the following succession of elementary transformations:

1. translation by $\alpha$
2. inversion
3. dilatation by $\rho$
4. rotation by $\theta$
5. translation by $\beta$.

#### III. Particular Möbius (5 pts)

Which is the Möbius transformation that makes a succession of:

1. a translation by $1+i$,
2. an inversion,
3. a dilation by $2\sqrt{2}$,
4. a rotation by $5\pi/4$
5. and a translation by 1.

Represent it on paper.

#### IV. Infinite Möbius (5 pts)

Study what happens with the mapping by the Möbius transform of the point that cancels the denominator. Are there points that are left untouched by the mapping?