Mathematical Lecture II
Handout 3: Exponentials, logarithms, roots, trigonometric functions, hyperbolics and their inverses.

Fabrice P. Laussy
*Departamento de Física Teórica de la Materia Condensada, Universidad Autónoma de Madrid**
(Dated: January 22, 2014)

Euler's formula is valid for complex arguments $e^{iz} = \cos z + i \sin z$, with $z \in \mathbf{C}$, as can be seen again by serie expansion:

$$e^{iz} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \tag{1}$$

and defining the sine an cosine of the complex variable z as the real and imaginary part of Eq. 1. This we can do since when z is real, we recover the usual trigonometric functions.

From this follows $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$. We also define $\tan(z) = \sin(z)/\cos(z)$.

Most properties extend to the complex realm, such as $\cos^2 z + \sin^2 z = 1$, $\cos(-z) = \cos(z)$, $\sin(-z) = -\sin(z)$, $\tan(-z) = -\tan(z)$, etc. Others break down, e.g., $|\sin(z)|^2 = \sin^2 x + \sinh^2 y$ and $|\cos(z)|^2 = \cos^2 x + \cosh^2 y$. This shows that the complex sine and cosine are unbounded.

Hyperbolic functions are defined as:

$$\cosh(z) = \frac{e^z + e^{-z}}{2}, \qquad \sinh(z) = \frac{e^z - e^{-z}}{2} \tag{2}$$

with also $\tanh(z) = \sinh(z)/\cosh(z)$, etc. We now have $\cosh^2 z - \sinh^2 z = 1$.

The link between trigonometric and hyperbolic functions is through complex numbers:

$$\sin(iz) = i\sinh(z), \quad \cos(iz) = \cosh(z), \quad \tan(iz) = i\tanh(z), \tag{3a}$$

$$\sinh(iz) = i\sin(z), \quad \cosh(iz) = \cos(z), \quad \tanh(iz) = i\tan(z). \tag{3b}$$

Notations exist that are important to know, such as $\sec(z) = 1/\cos(z)$, $\csc(z) = 1/\sin(z)$, $\cot(z) = 1/\cot(z)$ for the secant, cosecant, cotangent, etc. They also exist for the hyperbolic case where they can become monstruous, e.g., csch = 1/sinh. More common is sech = 1/ cosh.

More important than this terminology are inverse functions in the sense if $z = \sin(w)$, what is w as a function of z? The answer is $\arcsin(z)$. Because trigonometric and hyperbolic functions are all periodic, they are many-to-one; hence their inverses are necessarily multivalued. The most important ones are:

$$\arcsin(z) = -i\ln(iz + \sqrt{1-z^2}), \qquad \arccos(z) = -i\ln(z + i\sqrt{1-z^2}), \qquad \arctan(z) = -\frac{i}{2}\ln\left(\frac{1+iz}{1-iz}\right). \tag{4}$$

The derivatives of all these functions should also be known:

$$(\sin(z))' = \cos(z), \quad (\cos(z))' = \sin(z), \quad (\tan(z))' = \sec(z)^2, \tag{5a}$$

$$(\cot(z))' = -(\csc(z))^2, \quad (\sec(z))' = \sec(z)\tan(z), \tag{5b}$$

$$(\arcsin(z))' = 1/\sqrt{1-z^2}, \quad (\arccos(z))' = -1/\sqrt{1-z^2}, \quad (\arctan(z))' = 1/(1+z^2). \tag{5c}$$

A. Suggested readings

- Online encyclopedias, such as `http://en.wikipedia.org/wiki/Inverse_hyperbolic_function` or `http://mathworld.wolfram.com/InverseHyperbolicFunctions.html` and references therein.
- `http://laussy.org/wiki/MMII`

*Electronic address: fabrice.laussy@gmail.com

I. CONTINUOUS EXAMINATION

Dates by which to return the home exam (noted on 20):

1. 30 January
2. 13 February
3. 27 February
4. 13 March
5. 27 March
6. 10 April
7. 30 April

II. TO RETURN (BY "JANUARY 30" THE LATEST)

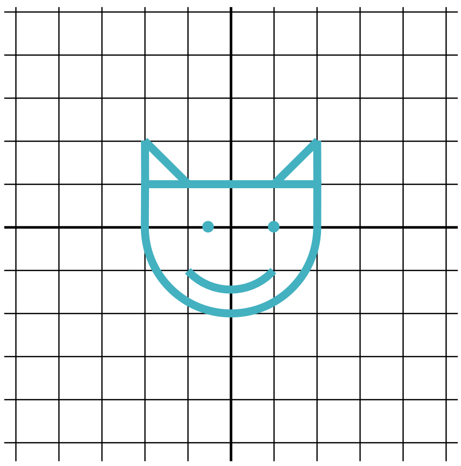
A. Part 1 (5 pts)

Compute the following expressions:

$$i^{12345}\,, \qquad \ln(\ln(i))\,, \qquad i+\cfrac{1}{i+\cfrac{1}{i+\cfrac{1}{i+\cfrac{1}{i+\cfrac{1}{i+1}}}}}\,.$$

B. Part 2 (5 pts)

Find the image of Arnold's cat by the transformation $z \to z^2$.



C. Problem (10 pts)

Find the complex numbers whose additive inverses and multiplicative inverses are the same.

> *First you have to figure out what an "additive inverse" and a "multiplicative inverse" is, if you don't know it. This is part of a problem, to understand what is asked. Sometimes this is the most difficult part, in particular as science is redundant with vague or even contradictory definitions and concepts.*