Mathematical Methods II
Handout 17. Series of Complex Numbers.

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A series is a sum of, typically, infinitely many terms. It therefore involves two concepts: that of a "sequence" of terms, that brings together the terms to be summed. And the limit of their sums. Let us start with defining the sequence (z_n). This is the notation for the collection of terms z_0, z_1, …, z_k with $k \in \mathbf{N}$. Such a sequence is called "convergent" if it has a limit. In symbols:

$$(\exists c \in \mathbf{C})(\forall \epsilon > 0)(\exists N \in \mathbf{N})(n > N) \Rightarrow (|z_n - c| < \epsilon)\,.$$

A sequence (z_n) of complex numbers $z_n = x_n + iy_n$ converges to $c = a + ib$ iff the sequence of real and imaginary parts of z_n converge to the real and imaginary part of c.

From there we are ready to introduce the notion of a series of complex numbers: given a sequence (z_n), we define another sequence (s_m) such that $s_m = \sum_{n=0}^m z_n$. Each term is called a partial sum and the sequence (s_m) itself is what is referred to as a "series". A convergent series is one who sequence of partial sums converge.

Theorem: If $\sum z_m$ converges, then $\lim_{m\to\infty} z_m \to 0$. *Proof:* Assuming that $\lim_{k\to\infty}\sum_k z_k$ converges, and calling s its limit, then since $z_m = \sum_{k=1}^m z_k - \sum_{k=1}^{m-1}$ for all m, taking the limit of both sides, and since we assumed the existence of the limit of $\sum_k z_k$, the limit of the difference is the difference of the limits and we find $\lim_{m\to\infty} z_m = \lim_{m\to\infty}\left(\sum_{k=1}^m z_k - \sum_{k=1}^{m-1} z_k\right) = s - s = 0$. Therefore, if z_m does not go to zero, it is immediate that the series made out of the corresponding sequence does not converge. Of course while $z_m \to 0$ is a necessary condition for convergence of the series, it is not sufficient. For instance, the harmonic series $\sum_{k=1}^\infty \frac{1}{k}$ diverges.

A series is called "absolutely convergent" if the series of absolute terms converge.

A series which is such that $\sum_k z_k$ converges but $\sum_k |z_k|$ diverges is called "conditionally convergent". For instance, the series $\sum_k \frac{(-1)^k}{k}$ converges but the sum of absolute terms diverges, as just mentioned.

It is not always easy to prove that a series converges. One chief difficulty is that the limit might not be done in the first place. In this case, one can use the powerful "Cauchy's Convergence Principle" which asserts that a series $\sum z_m$ is convergent iff for every $\epsilon > 0$, there exists N such that $|z_{n+1} + z_{n+2} + \cdots + z_{n+p}| < \epsilon$ for every $n > N$ and $p \in \mathbf{N}$.

Another useful property is that provided by the "comparison test": If $\sum b_n$ is a convergent series such that $|z_n| \le b_n$ for all n, then $\sum z_n$ converges absolutely.

This can be proved thanks to Cauchy's principle. Given that b_k converges, then by this principle, for any $\epsilon > 0$, there exists N such that $b_{n+1} + \cdots + b_{n+p} < \epsilon$ for every $n \ge N$ and for all $p \in \mathbf{N}$. From this and from the bounding of $|z_k|$ by the b_k, we have transported the Cauchy property to the $|z_k|$, i.e., $|z_{n+1}| + \cdots + |z_{n+p}| < \epsilon$, proving the convergence with the other implication of the Cauchy equivalence.

The geometric series is a good comparison series:

$$\sum_{m=0}^{\infty} q^m = \frac{1}{1-q} \tag{1}$$

if $|q| < 1$ and diverges otherwise.

"Ratio Test": If a series (z_n) has the property that for every n greater than N:

$$\left|\frac{z_{n+1}}{z_n}\right| \le q < 1 \tag{2}$$

the series converges absolutely; if for every $n > N$,

$$\left|\frac{z_{n+1}}{z_n}\right| \ge 1 \tag{3}$$

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then the series diverges. These will provide useful criteria for the power series that we will study later.

Proof: If 3 holds, then for all $n \geq N$, $|z_{n+1}| \geq |z_n|$ and therefore $|z_{n+1}| > |z_n|/2$ and therefore does not tend to zero, therefore impeding convergence of the series. On the other hand, if 2 holds, then $|z_{N+2}| \leq |z_{N+1}|q$, $|z_{N+3}| \leq |z_{N+1}|q^2$, that is, by recurrence:

$$|z_{N+p}| \leq |z_{N+1}|q^{p-1}\,, \tag{4}$$

from which we deduce:

$$|z_{N+1}| + |z_{N+2}| + \cdots \leq |z_{N+1}|(1 + q + q^2 + \cdots)\,, \tag{5}$$

which proves the absolute convergence of (z_n) from the comparison test (the sequence $(z_{N+1}q^n)$ is convergent, to $|z_{N+1}|/(1-q)$).

If the sequences of the ratios converge to a nonzero value (i.e., if zero is not an accumulation point, so that we can apply the ratio test past a large enough integer), it is convenient to estimate whether a series converge from the previous property:

Calling L the limit of the ratios:

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L\,.$$

Then:

- If $L < 1$, the series converges absolutely.
- If $L > 1$, the series diverges.
- If $L = 1$, the series may converge or diverge.

This is an immediate consequence of the ratio test applied to $k_n = |z_{n+1}/z_n|$, that can be made $< q$ for a n large enough.

A. Suggested readings

- `http://en.wikipedia.org/wiki/Series_(mathematics)`.
- On the meaning of providing a convergence to $1 + 2 + 3 + 4 + \cdots$, at `http://goo.gl/J7qHaV`.
- "The Euler-Maclaurin formula, Bernoulli numbers, the zeta function, and real-variable analytic continuation", Terence Tao, at `http://goo.gl/tET01`.

B. Exercises

1. Are the following series bounded? convergent? $z_n = (1+i)^{2n}/2^n$; $z_n = n^2 + i/n^2$ and $z_n = \sin(\frac{1}{4}n\pi) + i^n$.
2. If $z_i \to l$, what is the limit of $z_i + z_i^*$? convergent.
3. Which of these series are convergent:

$$\sum_{n=0}^{\infty}\frac{i^n}{n^2 - i}\,,\quad \sum_{n=1}^{\infty} n^2\left(\frac{i}{4}\right)^n\,,\quad \sum_{n=1}^{\infty}\frac{i^n}{n}\,. \tag{6}$$

C. Problems

1. Prove the counterpart of the ratio test: If a series (z_n) is such that for every $n > N$, $\sqrt[n]{|z_n|} \geq q < 1$ for some N, the series converges absolutely. If for infinitely many n, $\sqrt[n]{|z_n|} \geq 1$, it diverges.
2. Prove Cauchy's Convergence Principle for Series.