Mathematical Methods II
Handout 19. Uniform Convergence.

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Uniform convergence is convergence of functions as a whole $(\forall\epsilon>0)(\exists N\in\mathbf{N})(\forall z\in\mathcal{D})(n>N)\Rightarrow(|S_n(z)-f(z)|<\epsilon)$, rather than *pointwise*. The concept is important because several properties of the functions S_n, such as continuity and Riemann integrability, are transferred to the limit f if the convergence is uniform.

The Weierstrass M-test states that if a series $\sum_k u_k(z)$ is such that $|u_k(z)|\le M_k$ for all $z\in\mathcal{D}$, then if $\sum_k M_k$ converges, therefore $\sum_k u_k(z)$ converges uniformly. Let S_k a series of functions that are continuous on a domain $\mathcal{D}$ that contains the contour $\mathcal{C}$. If S_k converges uniformly to f on $\mathcal{D}$, then:

1. f is continuous on $\mathcal{D}$.
2. $\lim_{k\to\infty}\int_{\mathcal{C}} S_k(z)\,dz=\int_{\mathcal{C}} f(z)\,dz$

that is to say, we can interchange the integral sign and the limit.

A. Suggested readings

- *A history of analysis*, J. Hans Niels, AMS Bookstore (2003). Sec. "6.7 The Foundation of Analysis in the 19th Century: Weierstrass".

B. Exercises

1. Prove the uniform convergence of $\sum_{k=1}^{\infty} z^k/k^2$ on $\{z\,:\,|z|\le1\}$.
2. Where do these series converge uniformly? $\sum_{n=0}^{\infty}\left(\frac{n+2}{7n-3}\right)^n z^n$; $\sum_{n=0}^{\infty}(z+i)^{2n}/3^n$ and $\sum_{n=2}^{\infty}\binom{n}{2}(4z+2i)^n$.

C. Problems

1. Assume f_n and g_n converging uniformly to f and g, respectively. Show that f_n+g_n converges uniformly to $f+g$. Show that this is not compulsorily the case for f_ng_n.
2. Riemann's ζ function is defined as $\zeta(z)=\sum_{n=1}^{\infty}n^{-z}$. Show that ζ converges uniformly on $A=\{z\,:\,\mathrm{Re}(z)\ge2\}$.

D. Solutions to the Partial Examination

Complex Calculus

1. Since $i^2=-1$, $1+i-i^2=2+i$.
2. The sum reads $1-z+z^2$ which at $z=-i$ is $1+i+(-i)^2=i$.
3. The series $\sum_{k=0}^{\infty}z^k$ converges to $1/(1-z)$ [If we forgot it, we can find it again: let us call $S_N=\sum_{k=0}^{N}z^k$ the partial sum, then decomposing it as $S_{N+1}=\sum_{k=0}^{N}z^k+z^{N+1}$ on the one hand and and $S_{N+1}=1+\sum_{k=1}^{N+1}z^k=1+zS_N$ on the other hand, we find equating both terms $S_N+z^{N+1}=1+zS_N$, i.e., $S_N=(1-z^{N+1})/(1-z)$ which, incidentally, is also a result to know; the one we look for follow by taking the limit $N\to\infty$]. Therefore the series $\sum_{k=0}^{\infty}(-z)^k$ evaluates to $1/(1+z)$ which, for $z=i/2$ that is within the circle of convergence, is $1/(1+i/2)=2/(2+i)=2(2-i)/|2+i|=(4-2i)/5$.

Complex Plane

One could compute $|z_k-z_l|$ for all $k\le l$ and find which pairs provide the largest (question 4) and the smallest (question 5) distances.

By plotting the points on the complex plane, however:

4. It is obvious that the farthest points are z_0 and z_1.
5. The closest points could be z_0 and z_2 or z_1 and z_3, for which one can compute $|z_0-z_2|=|i-e^{i\pi/3}|$ and, which is easier by noting that they are aligned with the origin, $|z_1-z_3|=|e^{-i\pi/4}(1-\sqrt{2})|=\sqrt{2}-1$. The numerical evaluation of $|i-e^{i\pi/3}|$ can be made in several ways, e.g., by Pythagoras theorem, which yields $\sqrt{2}\sqrt{1-\sin(\pi/3)}=\sqrt{2-\sqrt{3}}$. To compare these numbers, we can compare their square since the square root is monotonous (hence $a>b\to\sqrt{a}>\sqrt{b}$) and

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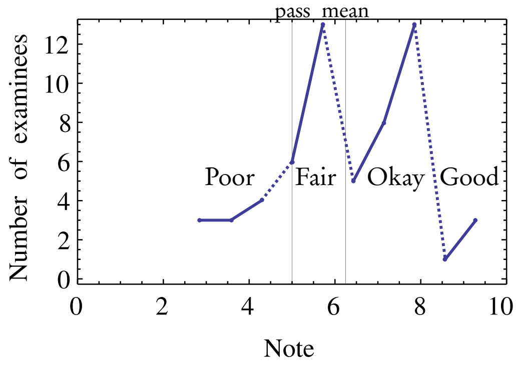

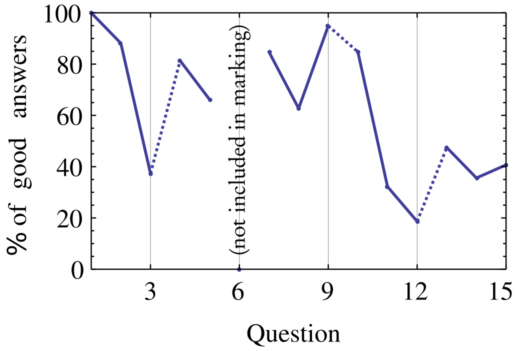

it is easy to estimate with pen and paper only that $(\sqrt{2}-1)^2 = 3-2\sqrt{2} \approx 3-2\times 1.41 \approx 0.18$ and $2-\sqrt{3} \approx 2-1.73 \approx 0.27$. This shows that z_1 and z_3 are the points closer to each others from the proposed set. The difference is ≈ 0.1 and should also be identifiable with a ruler.

6. The area is non zero and is clearly not complex, therefore the area is one of the three remaining expressions that can be evaluated graphically. We can also compute it by summing the area of sub-polygons, e.g., by decomposing into trapezoids delimited by the upper and lower points. Their area is easily calculated as the average of the heights times the basis, which, in the complex plane, yields:

$$\mathcal{A} = \sum_{i}^{N-1} \frac{\Im(z_{n+1}+z_n)\Re(z_{n+1}-z_n)}{2}\,. \tag{1}$$

On the given points, the result comes out as $(\sqrt{3}+2-\sqrt{2})/4 \approx 0.58$.

I. COMPLEX FUNCTIONS

7. Since $\exp(z) = \exp(x+iy) = \exp(x)\exp(iy) = \exp(x)(\cos(y)+i\sin(y))$, the real part is $e^x\cos(y)$.

8. Since $\ln(z) = \ln(re^{i\theta}) = \ln(r)+i\theta$, the imaginary part is θ, that is linked to the cartesian coordinates through $\tan\theta = y/x$ hence the imaginary part is $\arctan(y/x)$.

9. We compute the Cauchy–Riemann equations explicitly for $u(x,y) = 2x-y$ and $v(x,y) = ax+by$ and find $2 = b$ and $a = 1$.

II. CONFORMAL MAPPING (3PTS)

10. We want z such that $(z-i)/(z+i) = i$, i.e., $iz-1 = z-i$ which, factorizing z, yields $z(i-1) = 1-i$, that is, $z = -1$. We could as well have tried all the proposed solutions.

11. We know that the Möbius transform maps lines (and circles) into lines and circles. Since $0 \to -1$, $1+i \to (1-2i)/5$ and $1-i \to 1-2i$, which are not aligned, the image of $\theta = \pi/4$ will be that of a circle, i.e., of the type:

$$\left|\frac{z-i}{z+i} - A\right| = R^2\,, \tag{2}$$

with $A = x_0 + iy_0$ the center and R the radius. We have to determine the three values of the real numbers x_0, y_0 and R, which we can do by solving the set of equations for the points we have just evaluated:

$$\begin{cases} (x_0+1)^2 + y_0^2 = R^2 \\ (x_0-1)^2 + (y_0-2)^2 = R^2 \\ (x_0-1/5)^2 + (y_0+2/5)^2 = R^2 \end{cases} \tag{3}$$

with solution $x_0 = 0$, $y_0 = 1$ and $R = \sqrt{2}$, hence the image of the line with slope $\pi/4$ is the circle of centre i and radius $\sqrt{2}$.

12. The Möbius transform is a conformal mapping, i.e., it preserves angles, so that we do not need to actually compute the image of the curves but work out their angle directly. The images will intersect with the same angle. The angle between the line and the circle is given by the angle between the line and the tangent to the circle at the intersecting point. By elementary trigonometry, we find that $\theta = \pi/2 - \pi/3$ (where $\pi/2$ is the angle between the radius of the circle and the tangent to the circle, and $\pi/3$ is the angle between the radius and the line $y = \sqrt{3}/2$). The intersecting angle is therefore $\pi/6$ (or its complementary to π depending on the definition of the intersection).

III. COMPLEX INTEGRATION

13. The function $1/(z-i)$ is holomorphic on $\mathcal{C}$ and everywhere in the enclosed domain (since the pole is at i) which is outside the unit circle of center $-i$. Therefore the integral is zero.

14. The integral over $\mathcal{C}$ and along the segment $[-1,1]$ on the real axis is of an holomorphic funtion over a closed trajectory and is therefore zero. The integral over $\mathcal{C}$ is therefore minus that over $[-1,1]$, which can be computed by conventional methods:

$$\int_{-1}^{1} \exp(2z)\,dz = \frac{1}{2}e^{2z}\Big|_{-1}^{1} = \frac{e^2-e^{-2}}{2} = \sinh(2)\,.$$

Here we have used that $\exp(z)^2$ is $\exp(2z)$.

15. There is the pole at $z = 1$ in the domain circled by $\mathcal{C}$. Therefore we write the integral as:

$$\oint_{\mathcal{C}} \frac{4z^3+3z^2+2z+1}{z+1}\,\frac{1}{z-1}\,dz \tag{4}$$

where the first term $f(z)$ is holomorphic on $\mathcal{C}$ and the domain it contains. By Cauchy's integral formula for $n = 1$, we find $2i\pi f(1) = \oint_{\mathcal{C}} f(z)dz/(z-1) = 10i\pi$.